# Bresenham's Midpoint Algorithm

Lecture Set 1

CS5600 ***Computer Graphics***

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Spring 2005



## Line Characterizations

- Explicit: $y = mx + B$
- Implicit: $F(x, y) = ax + by + c = 0$
- Constant slope: $\frac{\Delta y}{\Delta x} = k$
- Constant derivative: $f'(x) = k$



## Line Characterizations - 2

- Parametric: $P(t) = (1-t)\, P_0 + t\, P_1$

  where, $P(0) = P_0\,; \quad P(1) = P_1$
- Intersection of 2 planes
- Shortest path between 2 points
- "Convex hull" of 2 discrete points



## Discrete Lines

- Lines vs. Line Segments
- What is a discrete line segment?
  - –This is a relatively recent problem
  - –How to generate a discrete line?

## "Good" Discrete Line

- No gaps in adjacent pixels
- Pixels close to ideal line
- Consistent choices; same pixels in same situations



## "Good" Discrete Line

- Smooth looking
- Even brightness in all orientations
- Same line for $P_0 P_1$ as for $P_1 P_0$
- Double pixels stacked up?



## Incremental Function Evaluation

- Recall $f(x_{i+1}) = f(x_i) + \Delta(x_i)$
- Characteristics
  - –Fast
  - –Cumulative Error
- Need to define $f(x_o)$



## Meeting Bresenham Line Criteria

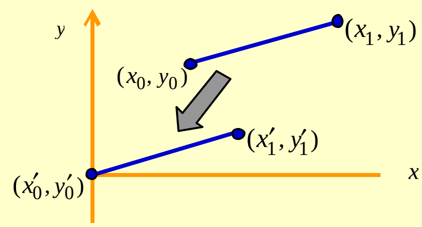
- $m = 0; \quad m = 1 \quad \Rightarrow$ trivial cases
- $(x_0, y_0) \neq (0,0) \quad \Rightarrow$ translate
- $0 > m > -1 \quad \Rightarrow$ flip about $x$-axis
- $m > 1 \quad \Rightarrow$ flip about $x = y$

## Case 0: Trivial Situations

- $m = 0 \quad \Rightarrow \quad$ horizontal line
- $m = 1 \quad \Rightarrow \quad$ line $\; y = x$
- Do not need Bresenham



## Case 1: Translate to Origin






## Case 1: Translate to Origin

- Move $(x_0, y_0)$ to the origin

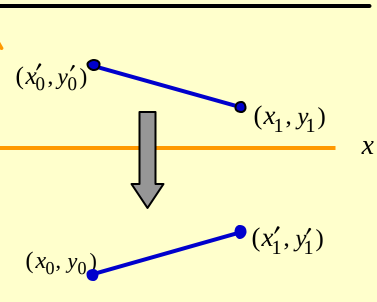
$$(x'_0, y'_0) = (0,0);$$

$$(x'_1, y'_1) = (x_1 - x_0, y_1 - y_0)$$

- Need only consider lines emanating from the origin.



## Case 2: Flip about x-axis



## Case 2: Flip about x-axis

- Suppose $0 > m > -1$,
- Flip about $x$-axis $(y' = -y)$:

$$(x'_0, y'_0) = (x_0, -y_0);$$
$$(x'_1, y'_1) = (x_1, -y_1)$$



## How do slopes relate?

$$m = \frac{y_1 - y_0}{x_1 - x_0};\quad m' = \frac{y'_1 - y'_0}{x_1 - x_0} \quad \text{by definition}$$

Since $y'_i = -y_i$, $m' = \frac{-y_1 - (-y_0)}{x_1 - x_0}$



## How do slopes relate?

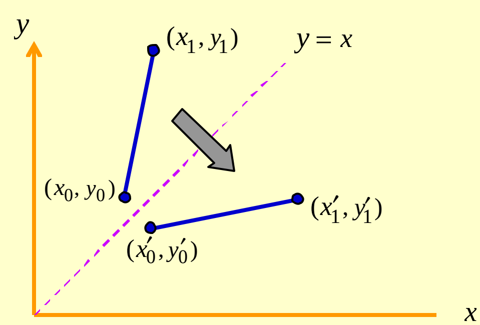
$$m' = -\frac{(y_1 - y_0)}{x_1 - x_0}$$

Or, $m' = -m$

$$\therefore\ 0 > m > -1 \ \Rightarrow\ 0 < m' < 1$$



## Case 3: Flip about line y=x



## Case 3: Flip about line y=x

$$y = mx + B,$$
swap $x \leftrightarrow y$ and prime them ,
$$x' = my' + B,$$
$$my' = x' - B$$



## Case 3: What is $m'$ ?

$$y' = \left(\frac{1}{m}\right) x' - B,$$
$$\therefore \quad m' = \left(\frac{1}{m}\right) \quad \text{and,}$$

$$m > 1 \quad \Rightarrow \quad 0 < m' < 1$$



## Enough to Generate Restricted Form

- Line segment in *first* quadrant with
$$0 < m < 1$$
- Let us proceed



## Two Line Equations

- Explicit: $y = mx + B$
- Implicit: $F(x, y) = ax + by + c = 0$

Define: $dy = y_1 - y_0$
$dx = x_1 - x_0$

Hence, $y = \left(\frac{dy}{dx}\right) x + B$

## From previous

we have $y = \left(\frac{dy}{dx}\right)x + B$

Hence, $\frac{dy}{dx}x - y + B = 0$

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## Relating Explicit to Implicit Eq's

Recall, $\frac{dy}{dx}x - y + B = 0$

Or, $(dy)x + (-dx)y + (dx)B = 0$

$\therefore\ F(x,y) = (dy)x + (-dx)y + (dx)B = 0$

where, $a = (dy);\quad b = -(dx);\quad c = B(dx)$

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## Investigate Sign of F

Verify that

$$F(x,y) \quad = \begin{cases} + & \text{below line} \\ 0 & \text{on line} \\ - & \text{above line} \end{cases}$$

Look at extreme values of $y$

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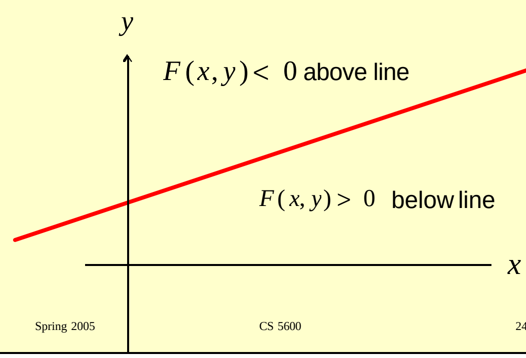
## The Picture

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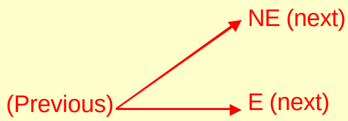
## Key to Bresenham Algorithm

"Reasonable assumptions" have reduced the problem to making a binary choice at each pixel:





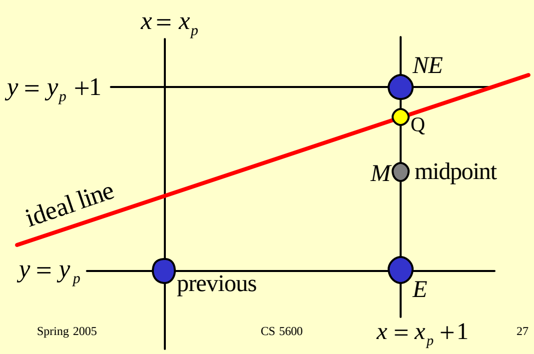

## Decision Variable (logical)

Define a logical *decision* variable $d$

- –linear in form
- –incrementally updated (with addition)
- –tells us whether to go *E* or *NE*

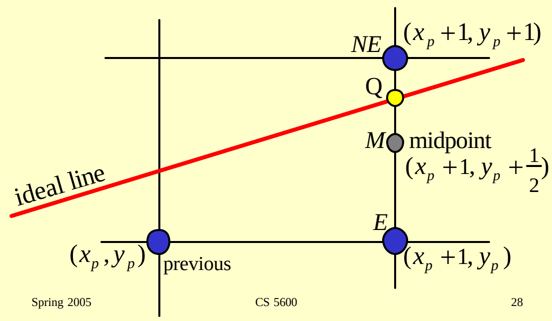


## The Picture






## The Picture (again)



## Observe the relationships

- Suppose Q is above *M*, as before.
- Then $F(M) > 0$, *M* is below the line
- So, $F(M) > 0$ means line is above *M*,
- Need to move *NE, increase y* value

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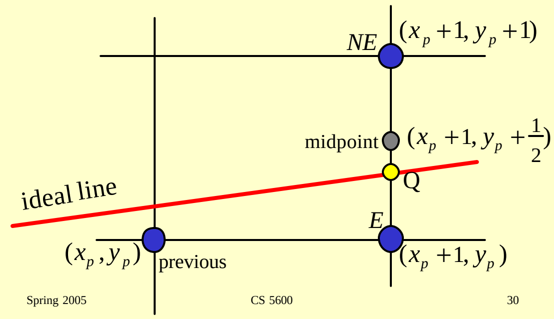
## The Picture (again)

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## Observe the relationships

- Suppose Q is below *M*, as before.
- Then $F(M) < 0$, *M* is above the line
- So, $F(M) < 0$ means line is below *M*,
- Need to move to *E; don't increase y*

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## *M* = Midpoint = $(x_p+1, y_p+\frac{1}{2})$

- Want to evaluate at *M*
- Will use an incremental *decision* $d$ variable $d = F(x_p+1, y_p+\frac{1}{2})$
- Let $d = a(x_p+1) + b(y_p+\frac{1}{2}) + c$

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## How will $d$ be used?

Recall $d = a(x_p + 1) + b(y_p + \frac{1}{2}) + c$

Therefore,

$$d = \begin{cases} > 0 & \Rightarrow NE \quad \text{(midpoint below ideal line)} \\ < 0 & \Rightarrow E \quad \text{(midpoint above ideal line)} \\ = 0 & \Rightarrow E \quad \text{(arbitrary)} \end{cases}$$

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## Case 1: Suppose *E* is chosen

- Recall $d_{old} = a(x_p + 1) + b(y_p + \frac{1}{2}) + c$
- $E \Rightarrow x \leftarrow x + 1; \quad y \leftarrow y,$

$$\therefore \quad d_{new} = F(x_p + 2, y_p + \frac{1}{2})$$

$$= a(x_p + 2) + b(y_p + \frac{1}{2}) + c$$

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## Case 1: Suppose *E* Chosen

$$d_{new} - d_{old} = \left( a(x_p + 2) + b(y_p + \frac{1}{2}) + c \right)$$

$$- \left( a(x_p + 1) + b(y_p + \frac{1}{2}) + c \right)$$

$$\boxed{d_{new} = d_{old} + a}$$

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## Review of Explicit to Implicit

Recall, $\frac{dy}{dx} x - y + B = 0$

Or, $(dy)x + (-dx)y + (dx)B = 0$

$$\therefore \quad F(x, y) = (dy)x + (-dx)y + (dx)B = 0$$

where, $a = (dy); \quad b = -(dx); \quad c = B(dx)$

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## Case 1: $d_{new} = d_{old} + a$

$\Delta_E \equiv$ increment we add if $E$ is chosen.

So $\Delta_E = a$. But remember that

$a = dy$ (from line equations).

Hence, $F(M)$ is not evaluated explicitly.

We simply add $\boxed{\Delta_E = a}$ to update $d$ for $E$

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## Case 2: Suppose *NE* chosen

Recall $d_{old} = a(x_p + 1) + b(y_p + \frac{1}{2}) + c$

and, $NE \Rightarrow x \leftarrow x + 1; \quad y \leftarrow y + 1,$

$$\therefore \quad d_{new} = F(x_p + 2, y_p + \frac{3}{2})$$

$$= a(x_p + 2) + b(y_p + \frac{3}{2}) + c$$

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## Case 2: Suppose *NE*

$$d_{new} - d_{old} =$$

$$= \left( a(x_p + 2) + b(y_p + \frac{3}{2}) + c \right)$$

$$- \left( a(x_p + 1) + b(y_p + \frac{1}{2}) + c \right)$$

$$\boxed{d_{new} = d_{old} + a + b}$$

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## Case 2: $d_{new} = d_{old} + a + b$

$\Delta_{NE} \equiv$ increment that we add if $NE$ is chosen.

So $\Delta_{NE} = a + b$. But remember that

$a = dy$, and $b = -dx$ (from line equations).

Hence, $F(M)$ is not evaluated explicitly.

We simply add $\Delta_{NE} = a + b$ to update $d$ for $NE$

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## Case 2: $d_{new} = d_{old} + a + b$

$\Delta_{NE} = a + b$, where $a = dy$, and $b = -dx$

means, we simply add $\Delta_{NE} = a + b$, i.e.,

$\Delta_{NE} = dy - dx$ to update $d$ for $NE$.



## Summary

- At each step of the procedure, we must choose between moving *E* or *NE* based on the sign of the $d$ decision variable
- Then update according to

$$d \leftarrow \begin{cases} d + \Delta_E, \text{ where } \Delta_E = dy, \text{ or} \\ d + \Delta_{NE}, \text{ where } \Delta_{NE} = dy - dx \end{cases}$$



## What is initial value of $d$?

- First point is $(x_0, y_0)$
- First midpoint is $(x_0 + 1, y_0 + \frac{1}{2})$
- Want initial midpoint value

$$d(x_0 + 1, y_0 + \frac{1}{2}) = F(x_0 + 1, y_0 + \frac{1}{2})$$



## What is initial value of $d$?

$$F(x_0 + 1, y_0 + \frac{1}{2}) = a(x_0 + 1) + b(y_0 + \frac{1}{2}) + c$$

$$= (ax_0 + by_0 + c) + \left(a + \frac{b}{2}\right)$$

$$= F(x_0, y_0) + \left(a + \frac{b}{2}\right)$$

## What is Initial Value of $d$ ?

Note, $F(x_0, y_0) = 0$, since $(x_0, y_0)$ is on line.

Hence,

$$F(x_0 + 1, y_0 + \frac{1}{2}) = 0 + a + \frac{b}{2}$$

$$= (dy) - \left(\frac{dx}{2}\right)$$

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## What is Initial Value of $d$ ?

Multiplyin g $F(x_0 + 1, y_0 + \frac{1}{2}) = (dy) - \left(\frac{dx}{2}\right)$

by 2 gives,

$$2F(x_0 + 1, y_0 + \frac{1}{2}) = 2(dy) - dx$$

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## What is the Effect of Multiplying by 2?

- Has the same zero set

$$2F(x, y) = 2(ax + by + c) = 0$$

- Changes the slope of the plane
- Rotates plane about the zero set line

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## What is Initial Value of $d$ ?

$$2F(x, y) = 2(ax + by + c) = 0$$

So first value of

$$d = 2(dy) - (dx)$$

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## More Summary

- Initial value $2(dy) - (dx)$
- Case 1: $d \leftarrow d + \Delta_E$, where $\Delta_E = 2(dy)$
- Case 2: $d \leftarrow d + \Delta_{NE}$,
  where $\Delta_{NE} = 2\{(dy) - (dx)\}$

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## More Summary

Choose $\begin{cases} E & \text{if } d \leq 0 \\ NE & \text{otherwise} \end{cases}$

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## Example

- Line end points:

  $(x_0, y_0) = (5,8); \quad (x_1, y_1) = (9,11)$

- Deltas: $dx = 4; dy = 3$

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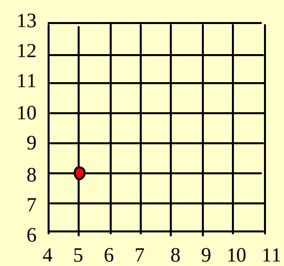
## Graph

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## Example ( $dx = 4; dy = 3$ )

- Initial value of

$$d(5,8) = 2(2y)-(dx)$$
$$= 6 - 4 = 2 > 0$$
$$d = 2 \Rightarrow NE$$

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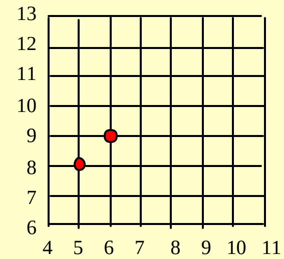
## Graph

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## Example ( $dx = 4; dy = 3$ )

- Update value of $d$
- Last move was $NE$, so

$$2d(6,9) = 2(dy\text{-}dx)$$
$$= 2(3 - 4) = -2$$
$$d = 2 - 2 = 0 \Rightarrow E$$

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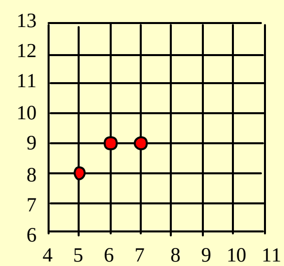
## Graph

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## Example ( $dx = 4; dy = 3$ )

- Previous move was *E*

$$d(7,9) = 2(dy)$$
$$= 2(3) = 6$$
$$d = 0 + 6 > 0 \Rightarrow NE$$

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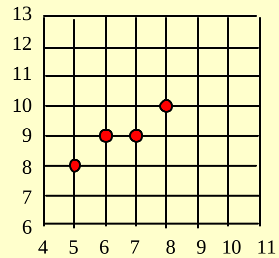
## Graph

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## Example ( $dx = 4; dy = 3$ )

- Previous move was $NE$, so

$$2d(8,10) = 2(dy - dx)$$
$$= 2(3 - 4) = -2$$
$$d = 6 - 2 = 4 \Rightarrow NE$$

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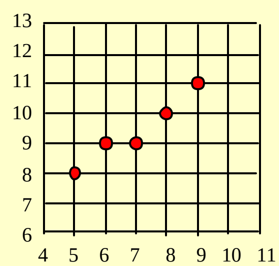
## Graph

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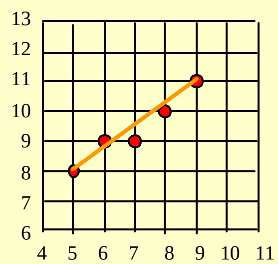
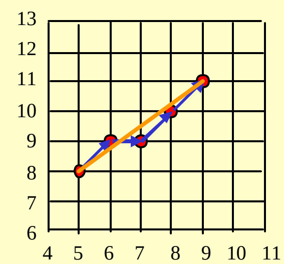
## Graph






## Graph



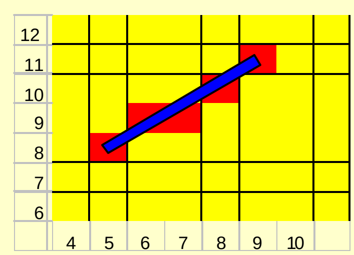



## More Raster Line Issues

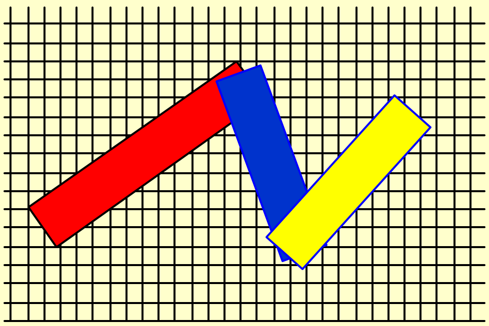
- Fat lines with multiple pixel width
- Symmetric lines
- End point geometry -- how should it look?
- Generating curves, e.g., circles, etc.
- Jaggies, staircase effect, aliasing...



## Pixel Space

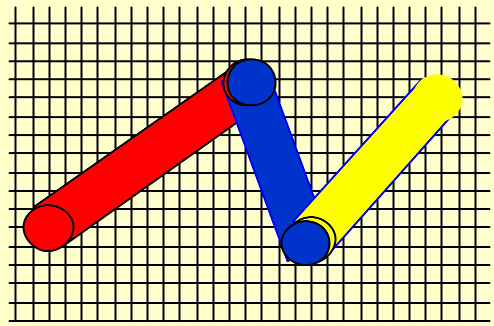


## Example



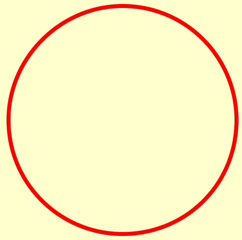


## Example





## Bresenham Circles





## The End of Bresenham's Algorithm

Lecture Set 1