ON THE EXISTENCE OF SOLUTIONS TO COMPLEMENTARITY FORMULATIONS OF CONTACT PROBLEMS WITH FRICTION

Mihai Anitescu* James F. Cremer† Florian A. Potra‡

Abstract

Complementarity formulations are a promising approach for solving dynamic multi-rigid-body contact problems. Here, we formulate the problem as a mixed linear complementarity problem (LCP). We introduce the notion of a disassemblable contact configuration and show that the matrix of the LCP for frictionless configurations is strictly copositive. As a result, in the case of dynamic friction, the matrix of the LCP for small friction coefficient values is also strictly copositive and the LCP again has a solution.

1 Introduction

The dynamic multi-rigid-body contact problem is concerned with predicting the accelerations and contact forces of rigid bodies in contact. Work in several research areas, robotics and virtual reality (VR) especially, has led to strong recent interest in this problem. Virtual reality applications, for instance, need improved physical believability; existing systems typically exhibit glaringly apparent physical inconsistencies that are due to a poor treatment of contact loss and the effects of multiple contacts between bodies.

The approach of classical dynamics simulation is to consider this problem as a piecewise differential algebraic equation (DAE, see Haug [7]) system, using the sign of the constraint force as a test to decide whether or not the corresponding contact will break. This approach implicitly assumes that the contact forces are continuous with respect to time. Another approach, initiated by Lötstedt in his Ph.D thesis (see also [8]), proceeds by formulating the multi-rigid-body contact problem as a complementarity problem. The latter approach works, in special cases, even if the contact forces are discontinuous.

For problems without friction, it is known that the LCP always has a solution [3, 13]. Unfortunately, for problems involving Coulomb friction, there is no guarantee that a solution exists, and several examples having no solutions have been demonstrated[8, 3]. Pang and Trinkle prove the existence of solutions to the contact problem in the

*Program in Applied Mathematics and Computational Sciences, The University of Iowa, Iowa City, IA 52242 (anitescu@math.uiowa.edu)

†Department of Computer Science, The University of Iowa, Iowa City, IA 52242 (cremer@cs.uiowa.edu)

‡Departments of Mathematics and Computer Science, The University of Iowa, Iowa City, IA 52242 (potra@math.uiowa.edu).

all-rolling case[10]. Recently, there has been interest in determining whether a contact problem has a solution for sufficiently small values of the friction coefficient [13].

In this paper, we first briefly describe the LCP model, and then give a necessary and sufficient condition under which the matrix of the LCP has a special property called strict copositivity. This property is maintained for small values of the friction coefficient and guarantees that the LCP has a solution.

The condition is the *disassemblability* of a frictionless configuration and essentially corresponds to existence of an external force that can break all contacts simultaneously. The notion is close to that of *strongly detachable fixture* from [11] but focuses on multibody systems rather than a unique manipulated object. In our work, we use techniques from linear programming to identify which part of the system is disassemblable and prove that a nondisassemblable system has a solution with infinite contact force. This approach is similar to the one in [12], where the notions of form closure, force closure, and strong force closure were analyzed in a quasi-static setting. The two earlier papers maintain a robotics perspective in which the torques of the manipulator joints are known and the contacts provide all of the kinematic constraints. Since we are concerned with simulating more general dynamical systems, we consider the effects of joint constraints when discussing disassemblability; joint constraints are explicitly included in the system rather than incorporated into special manipulator subsystems.

Although disassemblability is usually presented in conjunction with the stability of the assembly of a mechanical system, we do not address stability issues in this paper.

Note that analysis of disassemblability is not the goal of this paper. Instead, it provides means to characterize the existence of robustness of the LCP for small friction coefficients.

2 The assumptions and setting of the problem

2.1 The contact forces

Assume that we have two bodies in 3-dimensional space and that the system configuration can be represented by a vector $q \in R^{12}$ such that half of its coordinates account for translation and half for rotation. Assume that the bodies are in contact, as shown in Figure 1. Let N be the direction of the reaction force (thus, we will assume that $||N|| = 1$), T_1 and $T_2 = -T_1$ the directions of the friction force acting on the first and second body respectively, G_1 and G_2 the locations of the centers of mass of the bodies, and l_1 and l_2 the locations of the contact points in local coordinates (*i.e.*, with respect to G_1 and G_2). Let λ be the magnitude of the reaction force. Let H_1 and H_2 be the bodies' rotation matrices with respect to the world. We assume that the bodies are in relative motion; thus we consider dynamic friction only, and the direction of the friction is determined based on the direction of the relative velocity of the bodies. Also, we assume a Coulomb friction model, and let μ represent the friction coefficient.

Since the equations of motion are usually written with respect to the centers of mass of the bodies involved in the simulation, the force must be expressed with respect to the centers of mass. The reaction force λN at the contact point generates the force $-\lambda N$ and the torque $-H_1 l_1 \times \lambda N = -\lambda H_1 l_1 \times N$ with respect to the center of mass G_1, as well as the force λN and the torque $\lambda H_2 l_2 \times N$ with respect to the center of mass G_2. Similarly, the friction force generates a force $\mu\lambda T_1$ and a torque $\mu\lambda H_1 l_1 \times T_1$ with respect to the center of mass G_1, as well as a force $\mu\lambda T_2$ and a torque $\mu\lambda H_2 l_2 \times T_2$ with respect to the center of mass G_2. Anticipating the generalized setting, we introduce the following vectors

$$R = \begin{pmatrix} -N \\ -H_1 l_1 \times N \\ N \\ H_2 l_2 \times N \end{pmatrix} \tag{2.1}$$

and

$$F = \begin{pmatrix} T_1 \\ H_1 l_1 \times T_1 \\ T_2 \\ H_2 l_2 \times T_2 \end{pmatrix}. \tag{2.2}$$

Then we can say that the 12-dimensional vectors λR and $\mu\lambda F$ represent the generalized reaction and friction forces for this two-body configuration.

2.2 The kinematic contact constraint

To prevent interpenetration between bodies, we require that $d \geq 0$, where d is a distance measure as defined in [1]. This d is defined such that when there is no interpenetration d is positive and corresponds to the interbody distance. When there is interpenetration, d is negative, and when objects are in perfect contact, d is zero. In the case where $d = 0$ and we have a point contact (as in Figure 1) we require a kinematic constraint that, essentially, prohibits translation in the direction of N. To express that restriction in a workable fashion, it is customary to work in a Newton-Euler setting. Let v_1, ω_1, v_2, and ω_3 be the translational and rotational velocities of the first and second bodies, expressed in world coordinates. Then it can be shown (see, for instance, [1]) that

$$\dot{d} = (v_2 - v_1)^T N + (H_2 l_2 \times N)^T \omega_2 - (H_1 l_1 \times N)^T \omega_1. \tag{2.3}$$

For the second derivative, we get

$$\ddot{d} = (\dot{v}_2 - \dot{v}_1)^T N + (H_2 l_2 \times N)^T \dot{\omega}_2 - (H_1 l_1 \times N)^T \dot{\omega}_1 + \gamma, \tag{2.4}$$

where γ is a number that depends on the bodies' velocities and positions (but not the accelerations). Let

$$a = \begin{pmatrix} \dot{v}_1 \\ \dot{\omega}_1 \\ \dot{v}_2 \\ \dot{\omega}_2 \end{pmatrix} \tag{2.5}$$

be the acceleration vector. Then, using R from (2.1), we can rewrite (2.4) as

$$\ddot{d} = R^T a + \gamma. \tag{2.6}$$

2.3 The complementarity condition

Assume that the two bodies have been in contact for some time. Therefore $d = 0$ and $\dot{d} = 0$. Since we require that $d \geq 0$ it follows that we must ensure $\ddot{d} \geq 0$ in order to avoid interpenetration in a subsequent instant. If the forces have the directions shown in Figure 1, it follows that $\lambda \geq 0$ since the two bodies may push but not pull on each other (*i.e.* the contact can only sustain compressive force). This is the main difference between contact and traditional joint constraints which can assume both positive and negative reaction forces. Thus, contact constraints are sometimes called *unilateral* constraints, while joint constraints are termed *bilateral* constraints.

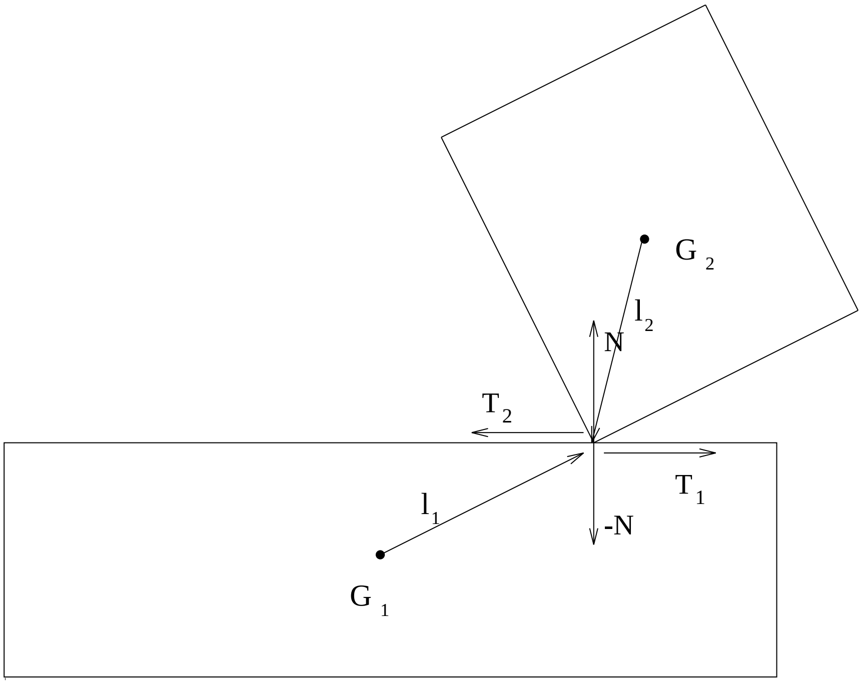

Figure 1: Generic contact situation between two bodies. N is the direction of the reaction force. T_1 and $T_2 = -T_1$ are the directions of the frictional force. G_1 and G_2 are the locations of the centers of mass, l_1 and l_2 are the locations of the contact points with respect to the centers of mass.

Assume now that $\ddot{d} > 0$. This means that in the following instant the contact between the two bodies will break. As a result, the mutual force must vanish, yielding $\lambda = 0$. Combining the two cases gives

$$\ddot{d}\lambda = 0. \tag{2.7}$$

Two nonnegative variables that satisfy such a condition are said to be complementary variables.

2.4 The contact problem for two bodies

For the subsequent derivations we define $s = \ddot{d}$. Then by (2.7) we must have $s^T\lambda = 0$ and by (2.6) we must have $s = R^T a + \gamma$, where both s and λ are nonnegative. Now we can write Newton's law, which connects acceleration and forces:

$$Ma = Q + \lambda R + \mu\lambda F, \tag{2.8}$$

where M is the generalized mass matrix for the two bodies, a block diagonal matrix composed of 3-by-3 blocks. The blocks corresponding to $\dot{v}_1$ and $\dot{v}_2$ are diagonal having the corresponding masses as entries. The blocks corresponding to $\dot{\omega}_1$ and $\dot{\omega}_2$ are the inertia matrices written in world coordinates. Q is the sum of the external and Coriolis forces, the latter being computed once the positions and velocities are known. Finally λR and $\mu\lambda F$ are the contact reaction and friction forces.

The main problem in dynamics is to find the accelerations given the positions and velocities of the bodies involved in the contact. This problem can be restated as mixed linear complementarity problem of the form: Find (a, s, λ) such that

$$\begin{pmatrix} 0 \\ s \end{pmatrix} = \begin{pmatrix} M & -(R+\mu F) \\ R^T & 0 \end{pmatrix} \begin{pmatrix} a \\ \lambda \end{pmatrix} + \begin{pmatrix} -Q \\ \gamma \end{pmatrix} \tag{2.9}$$

$$s^T\lambda = 0; \quad s, \lambda \geq 0 \tag{2.10}$$

2.5 The general setting of the contact problem

Suppose that we have a system of several bodies along with a set of m contacts between the bodies. For each pair of bodies in contact use an analysis similar to that above. Each contact i, $1 \leq i \leq m$ will generate an s_i (slack) variable, a row R_i^T, and a generalized force $\lambda_i(R_i + \mu F_i)$. When the friction coefficients are not the same for each i, the F_i's should be scaled accordingly. For completeness, assume that the system has, in addition to contact constraints, l joint-type constraints defined by the map $\Theta(q)$ taking the configuration space to R^l. Here by q we mean the generalized position of the system of bodies, which can be written for instance as the concatenation of the translation and rotation coordinates of the bodies. We will also denote by $\dot{q}$ and $\ddot{q}$ the generalized positions and velocities. The constraints can be written as $\Theta(q) = 0$. For details concerning joint constraints and forces see, for example, [6].

By writing the joint and contact constraints together with Newton's law and the appropriate constraint forces we get the following equation

$$\begin{pmatrix} 0 \\ 0 \\ s \end{pmatrix} = \begin{pmatrix} M & -\Theta_q & -(R+\mu F) \\ \Theta_q^T & 0 & 0 \\ R^T & 0 & 0 \end{pmatrix} \begin{pmatrix} \ddot{q} \\ \nu \\ \lambda \end{pmatrix} + \begin{pmatrix} -Q \\ \Gamma_1 \\ \Gamma_2 \end{pmatrix}. \tag{2.11}$$

where $s_i = \ddot{d}_i$, $1 \leq i \leq m$ is the vector of contact accelerations. Q is the sum of input forces and the Coriolis forces, Γ_1 and Γ_2 are terms involving the velocities $\dot{q}$. R is the matrix whose columns are the R_i's, and F is the matrix whose columns are the F_i's. M is the mass matrix which is positive definite. The dimension of M is $n \times n$, where n is the numbers of degrees of freedom of the system if there were no constraints (in three dimensions $n = 6k$, where k is the number of the bodies in the system). $\nu\Theta_q$ is a vector of the joint reaction forces. We can assume without loss of generality that Θ_q has full row rank, for otherwise we can reduce the dependent rows and deal with a smaller set of ν's.

For each contact the complementarity relation (2.7) must hold. Thus, $s_i\lambda_i = 0,\ 1 \leq i \leq m$. By adding up the relations (2.7) for all the contact constraints, we get

$$s^T\lambda = 0; \quad s, \lambda \geq 0. \tag{2.12}$$

The relations (2.12) and (2.11) determine a mixed linear complementarity problem [4].

3 Disassemblability

3.1 Definition of disassemblability

It is known that in the frictionless case ($\mu = 0$) the former linear complementarity problem has very good properties. For instance, the matrix on the left hand side of (2.11) becomes positive semidefinite, and therefore the mixed LCP is equivalent to a traditional monotone LCP (in which no equality constraints are involved) [2, 14]. For such an LCP, if nonnegative s and λ exist so that (2.11) is satisfied, then a complementary solution (satisfying (2.12)) exists [4]. Moreover it is easy to see that the accelerations $\ddot{q}$ and the slacks s are unique for any solution of the LCP. In this section, we will show that under mild assumptions the friction problem inherits some of the good behavior of the frictionless problem for a sufficiently small μ. We will consider first the frictionless case and discuss later the impact of having a nonzero friction coefficient μ.

It is useful to rewrite (2.11) such that the accelerations do not appear in the new formulation. We solve for the accelerations in the first row of (2.11) and then substitute for them in the other rows. We then get an equivalent mixed LCP

$$\begin{pmatrix} s \\ 0 \end{pmatrix} = \begin{pmatrix} R^TM^{-1}R & R^TM^{-1}\Theta_q \\ \Theta_q^TM^{-1}R & \Theta_q^TM^{-1}\Theta_q \end{pmatrix} \begin{pmatrix} \lambda \\ \nu \end{pmatrix} + \begin{pmatrix} R^TM^{-1}Q + \Gamma_2 \\ \Theta_q^TM^{-1}Q + \Gamma_1 \end{pmatrix}. \tag{3.1}$$

$$s^T\lambda = 0; \ s, \lambda \geq 0. \tag{3.2}$$

We would like to determine if there exists an external force Q such that (3.1) (hence (2.11)) is satisfied and, at the same time, the slacks are strictly positive. More specifically, we want to determine whether the polytope

$$\mathcal{P}_1 = \left\{s \geq 0 \middle| \ \exists \chi \in R^n, \ s = R^T\chi; \ \Theta_q^T\chi = 0\right\} \tag{3.3}$$

has an interior point $s_0 > 0$ with a corresponding χ_0. When it does, let $\nu = -(\Theta_q^TM^{-1}\Theta_q)^{-1}\Gamma_1$, which is well-defined since M is positive definite and Θ_q has full row rank. And let $Q = tM\chi_0$ for t such that $tR^T\chi_0 + \Gamma_2 + R^TM^{-1}\Theta_q\nu = ts_0 + \Gamma_2 + R^TM^{-1}\Theta_q\nu > 0$. Then, (λ, ν, s) with $\lambda = 0$, ν defined as above and $s = ts_0 + \Gamma_2 + R^TM^{-1}\Theta_q\nu > 0$ is a solution of the LCP (3.1). Hence applying the force Q effects the immediate breakage of all contacts. The system can thus be disassembled with respect to the contact constraints (while remaining consistent with respect to the joint-type constraints). For this reason, a configuration for which the polytope $\mathcal{P}_1$ has an interior point (*i.e.*, (3.1) has a solution with strictly positive s's) is called *disassemblable.*

For s_0, ν, and Q as defined above, $s_0 > 0$, $\lambda = 0$ and ν constitute a solution of the LCP (3.1). If the system would have had friction, the only change would be that R would be replaced by $R+\mu F$ everywhere it appears to the right of M^{-1} in (3.1). It is easy to see that s_0, $\lambda = 0$, ν would be a solution of the problem with friction as well. Thus, both the friction and the frictionless configurations can be disassembled by the Q chosen above. Hence, friction is irrelevant for the disassemblability property.

Note, however, that disassemblability is perhaps less well defined defined in the friction case since for the LCP with friction, there is no guarantee, even when a solution exists, that the s's are unique. This leaves opened the possibility that a different solution exists for the same Q that does not disassemble the configuration. Below, we will consider, without loss of generality, just the case $\mu = 0$ when discussing disassemblability, but will account for the case $\mu \neq 0$ when looking for solutions of the frictional versions of the LCP (3.1).

3.2 Characterization of nondisassemblability

Assume that a frictionless contact configuration is not disassemblable. Then, the polytope $\mathcal{P}_1$ has no interior point and there is no χ exists such that:

$$s = R^T\chi > 0 \tag{3.4}$$

$$\Theta_q^T\chi = 0. \tag{3.5}$$

Now, let I be the set of indices i for which $\exists s \in \mathcal{P}_1$ with positive ith component ($s_i > 0$). For each $i \in I$, let s^i be a point satisfying the above property (i.e., $s_i^i > 0$) and let $\chi^i \in R^n$ the corresponding χ from 3.4. It follows that if we take $s^{pos} = \sum_{i\in I} s^i$, then $s_i^{pos} > 0, \forall i \in I$. And, clearly, $s^{pos} \in \mathcal{P}_1$. Thus if the only contacts were those corresponding to the inequalities i for $i \in I$, we would have an interior point. The "problem" index set is thus the complement of I with respect to the total number of inequalities and we call it $\overline{I}$. Following its definition we have that for any $s \in \mathcal{P}_1$, $s_i = 0$ for all $i \in \overline{I}$. Let χ^{pos} satisfy $\Theta_q^T\chi^{pos} = 0$ and $s^{pos} = R^T\chi^{pos}$. Such a χ^{pos} can be obtained by taking $\chi^{pos} = \sum_{i\in I}\chi^i$. Also, let W be the matrix consisting of those columns of R whose indices are in $\overline{I}$. Consider now the polytope:

$$\mathcal{P}_2 = \left\{w \geq 0 | \quad \exists\chi \in R^n \quad w = W^T\chi, \quad \Theta_q^T\chi = 0\right\}. \tag{3.6}$$

Lemma 3.1 $\mathcal{P}_2 \equiv \{0\}$.

Proof Assume there is a $w \in \mathcal{P}_2$ containing some positive elements. Then, there must be a corresponding χ_1 such that $w = W^T\chi_1$ and $\Theta_q^T\chi_1 = 0$. By taking $\tilde{\chi} = \chi_1 + \delta\chi^{pos}$, with δ sufficiently big, it follows that

$$\Theta_q^T\tilde{\chi} = 0 \tag{3.7}$$

$$R^T\tilde{\chi} \geq 0. \tag{3.8}$$

Thus, we have a point $s^+ = R^T\tilde{\chi} \geq 0$. By construction, we have that $s_i^+ > 0$ for $i \in I$, and $s_j^+ > 0$ for some $j \in \overline{I}$, (where the corresponding component of w is positive). But this contradicts the definition of I, and so the proof is complete. □

Lemma 3.2 *There is a vector $\eta \in R^m$ such that $\eta_i = 0$ for $i \in I$ and $\eta_i > 0$ for $i \in \overline{I}$ and a vector $\zeta \in R^l$ such that*

$$R\eta + \Theta_q\zeta = 0 \tag{3.9}$$

Proof Without loss of generality, we may assume that $\Theta_q^T = [I, -N]$. Let $W^T = [A_1, A_2]$ be the corresponding partition of W^T and $\chi = [\chi_1^T, \chi_2^T]^T$ the partition of χ, where $\chi \in R^n$. Then $\chi_1 = N\chi_2$ for χ in accordance with the definition of the polytope 3.6. The fact that $\mathcal{P}_2 \equiv \{0\}$ can be now written as

$$A_1 N\chi_2 + A_2\chi_2 \geq 0 \Rightarrow A_1 N\chi_2 + A_2\chi_2 = 0 \tag{3.10}$$

and thus the system

$$A_1 N x_2 + A_2 x_2 \geq 0, \quad A_1 N x_2 + A_2 x_2 \neq 0 \tag{3.11}$$

has no solution. By Stiemke's (alternative) theorem [4, Thm.2.7.12] it follows that there is an $\eta' > 0$ such that $A_2^T\eta' + N^T A_1^T\eta' = 0$. By taking $\zeta = -A_1^T\eta'$, it follows immediately that $W\eta' + \Theta_q\zeta = 0$. But now by taking the entries of η to be the same as those of η' for the indices in $\overline{I}$ and zero otherwise, we obtain (since W is the restriction of R on the index set $\overline{I}$) that $R\eta + \Theta_q\zeta = 0$ and $\eta_i > 0$ for $i \in \overline{I}$. □

Assume that $(\ddot{q}, \lambda, \nu)$ is a solution of (2.11). Then clearly $(\ddot{q}, \lambda + t\eta, \nu + t\zeta)$ is also a solution of (2.11) for any $t > 0$. This means that the bodies can assume an infinite reaction force (when taking $t \to \infty$). This behavior is likely to cause problems when introducing friction; since friction forces are proportional to the reaction forces, an indeterminacy in reaction forces can yield indeterminacy in the friction forces and, therefore, in the accelerations as well.

The previous development also makes a new point. Since there is a part of the system that cannot be disassembled (contacts for which the corresponding slacks s_i are always zero) we can treat those particular contacts as bilateral constraints. Thus, assume that W, the columns of R corresponding to $\overline{I}$, is taken into account as columns of Θ_q, and the reaction force splits as $\nu = [\nu_1, \nu_2]$, where ν_2 represents the reactions in the old bilateral constraints and ν_1 represent the reaction force in the nondisassemblable contact constraints. This new configuration is disassemblable, and the positive entries of s^{pos} are the corresponding positive slacks. Now let η^{pos} be the positive entries of η from the previous lemma corresponding to the index set $\overline{I}$. And let (λ, ν, s) with $\nu = [\nu_1, \nu_2]$ be a solution of (2.11) corresponding to the new R and Θ_q. Since the nondisassemblable constraints are treated as bilateral constraints, there is no guarantee that ν_1 is positive. By our previous developments $(\lambda, \nu_1 + t\eta^{pos}, \nu_2 + t\zeta)$ is also, for any t, a solution of the LCP (2.11) with the new data. But for a sufficiently big t we get that $\nu_1 + t\eta^{pos} > 0$, and therefore we recover a solution to the initial LCP. Hence we have proved the following result.

Theorem 3.3 *A frictionless contact problem can always be cast as a contact problem with disassemblable contact constraints, consistent with the bilateral constraints.*

3.3 Disassemblability and copositivity

Since we may assume that Θ_q has full rank it follows that $\Theta_q^T M^{-1}\Theta_q$ is invertible and we can solve for the ν's from the joint equations (the bottom rows) in (3.1). The result is a standard LCP of the form

$$s = (R^T M^{-1} R - R^T M^{-1}\Theta_q(\Theta_q^T M^{-1}\Theta_q)\Theta_q^T M^{-1} R)\lambda + R^T M^{-1} Q + \Gamma_2 - \\ R^T M^{-1}\Theta_q(\Theta_q^T M^{-1}\Theta_q)^{-1}(\Theta_q^T M^{-1} Q + \Gamma_1) \tag{3.12}$$

$$s^T\lambda = 0; \; s, \lambda \geq 0. \tag{3.13}$$

A *strictly copositive* matrix is a matrix M with the property $x^T M x > 0$ for any $x \geq 0, x \neq 0$ [4, Def. 3.8.1]. If the matrix of an LCP is strictly copositive then the

LCP will have a solution for any free term and a special algorithm, Lemke's algorithm [4, Sec.4.4], is guaranteed to find it.

Theorem 3.4 *The matrix of the LCP (3.12) is strictly copositive if and only if the contact configuration is disassemblable.*

Proof The matrix in (3.12) is positive semidefinite by construction. Therefore the LCP matrix M_{LCP} is not strictly copositive if and only if there is $\lambda \geq 0, \lambda \neq 0$ such that $\lambda^T M_{LCP} \lambda = 0$. If we take $\nu = -(\Theta_q^T M^{-1} \Theta_q)^{-1} (\Theta_q^T M^{-1} R) \lambda$ and use (3.12) we can write

$$\lambda^T M_{LCP} \lambda = (\lambda \ \nu) \begin{pmatrix} R^T \\ \Theta_q^T \end{pmatrix} M^{-1} (R \ \Theta_q) \begin{pmatrix} \lambda \\ \nu \end{pmatrix}. \tag{3.14}$$

If $\lambda^T M_{LCP} \lambda = 0$ it follows that $R\lambda + \Theta_q \nu = 0$ since M^{-1} is positive definite. But by the argument in the previous section this can happen if and only if the configuration is not disassemblable, which proves the claim. □

Corollary 3.5 *If a multibody contact configuration with frictionless joints is disassemblable or if the nondisassemblable part is frictionless then, for a sufficiently small friction coefficient μ, the corresponding LCP has a solution that can be found by Lemke's method.*

Proof By Theorem 3.3 we know that the frictionless problem can be recast as an LCP problem with disassemblable contacts. From Theorem 3.4 we know that the LCP matrix M_{LCP} is strictly copositive. We can introduce friction in any configuration, by simply replacing R with $R + \mu F$ in (3.12) any time it appears multiplied on the right of M^{-1}. This means that $M_{LCP,\mu} = M_{LCP} + \mu M_F$. Since M_{LCP} is a strictly copositive matrix , so is $M_{LCP,\mu}$ for a sufficiently small μ which proves the claim. □

3.4 Aspects of nondisassemblability

One example of a typical nondisassemblable configuration is the physical realization of a translational joint such as the one shown in Figure 2. The slider is assumed to be in contact with the bar on four sides. Clearly, such a configuration is not disassemblable. Assume that we have friction at the contacts and that we are pulling the slider with a horizontal force Q (we neglect gravity and assume the bar is fixed). By Lemma 3.2 there is a set of contact forces that cancel each other. Any positive multiple of those contact forces would also satisfy (2.11) if the motion were frictionless. It follows that the slider can acquire any acceleration a between 0 and Q/m if the slider were initially at rest, where m is the mass of the slider (for each acceleration in that range we choose the reaction force to give the total friction $Q - ma$). Therefore, nondisassemblable configurations have a very high indeterminancy. In practice, such translational joints are lubricated and a simpler model (and more accurate for small reaction forces) would be to have the friction proportional to the velocity, and thus on the free term of the LCP. Technically, the LCP to be solved is the same as the frictionless one. And since, by the previous developments, the nondisassemblably part of the configuration can be treated as a joint, one does not have to worry about determining positive forces.

A more realistic model would ensure some space between the slider and the bar. This case could still be modeled as an LCP, but simulating such a configuration would get very difficult because of the many collisions that would need to be accounted for.

Thus, a more realistic (and robust) way of modeling friction in nondisassemblable configurations is to consider viscous friction, in which case the LCP for the nondisassemblable part is the same as for frictionless joints.

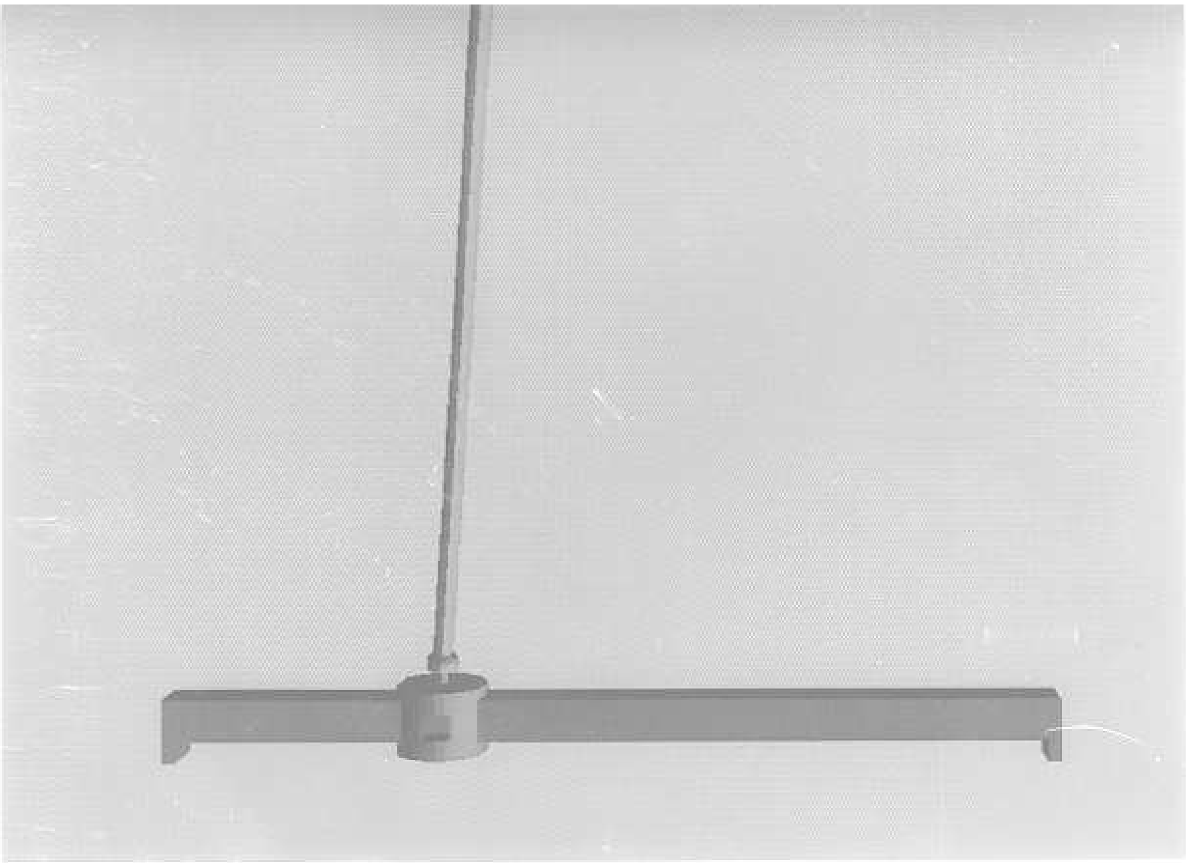

Figure 2: A translational joint

In general, determining if a configuration is disassemblable means to determine wether the corresponding polytope has an interior point. But the disassemblability of most configurations can be determined by inspection, for nondisassemblability by accident is very unusual. Therefore, most contact configurations have a solution for small values of the friction coefficient. Determining the value of μ for which we still have a strictly copositive problem is more difficult and there is no guarantee that we can determine a solution of the LCP if μ has a relatively high value. Of course, some bounds on μ can be found in terms of the operator norms of certain matrices, but at this stage, they do not seem to be very relevant. In [9] there is a bound for μ on a problem similar to (2.11). But the bound is found under the assumption that the matrix in that particular setting (a different matrix than ours) is diagonally dominant with positive entries on the diagonals. Translating the somewhat restrictive assumption to fit our setting of the problem does not seem likely to prove especially valuable. Our goal in the present paper was to develop general conditions on the existence of solutions in the presence of friction; we leave the problem of determining of a good bound on the friction coefficient for future research.

4 Conclusion

In this paper, we attempted to examine the connection between the structure of contact and the properties of the corresponding LCP. We introduced the notion of disassemblable configuration and showed that it is equivalent to an LCP with a strictly copositive contact matrix. The conclusion is that a disassemblable contact configuration is robust with respect to small variations in friction. The problem of deciding if a configuration is disassemblable is the problem of finding an interior point, and this is solvable in polynomial time. The algorithm for finding the interior point can also be modified to identify any nondisassemblable parts. In practice, one can usually tell by inspection whether or not a mechanism is disassemblable.

Moreover, we showed that nondisassemblable frictionless configurations can be reduced to disassemblable configurations. In addition, if the nondisassemblable parts are modeled with viscous (instead of dry friction) the reduction is valid for the friction case as well.

Acknowledgements

We thank the reviewers for their insightful comments. Partial support for this work was provided by Office of Naval Research grant N00014-96-1-0269.

References

[1] M. Anitescu, J. F. Cremer, and F. A. Potra. Formulating 3D Contact Dynamics Problems. To appear in Mechanics of Structures and Machines, 22(4), 1996.

[2] M.Anitescu, G.Lesaja and F.A.Potra. Equivalence between different formulations of the linear complementarity problem. University of Iowa Technical Reports in Computational Mathematics, Number 71, June 1995. Submitted for publication.

[3] D. Baraff. Issues in Computing Contact Forces for Non-Penetrating Rigid Bodies. *Algorithmica* 10:292-352,1993.

[4] R.W.Cottle, J.S.Pang, and R.E.Stone. *The Linear Complementarity Problem.* Academic Press, 1992.

[5] M.P. do Carmo. *Differential Geometry of Curves and Surfaces.* Prentice-Hall,Englewood Cliffs,NJ,1976.

[6] E.J.Haug. *Computer aided kinematics and dynamics of mechanical systems.* Allyn and Bacon, 1989.

[7] E.J.Haug, S.C.Wu and S.M.Yang. Dynamic mechanical systems with Coulomb friction, stiction, impact and constraint addition-deletion I: Theory. *Mechanisms and Machine Theory*, 21(5):407-416,1986.

[8] P.Lötstedt. Mechanical systems of rigid bodies subject to unilateral constraints.*SIAM Journal of Applied Mathematics*, 42(2):281-296,1982.

[9] Z. Liu. Convergence analysis of projected Jacobi method for contact force computation with friction. *Proceedings of the First ACM Workshop on Simulation and Interaction in Virtual Environments*, Iowa City, 320–324, July, 1995.

[10] J.-S. Pang. and J. C. Trinkle. Complementarity Formulations and Existence of Solutions of Multi-Rigid-Body With Coulomb Friction. To appear in *Mathematical Programming*.

[11] J.M.Schimmels and M.A.Peshkin. Admittance Matrix Design for Force–Guided Assembly. *IEEE Transactions on Robotics and Automation*, 8(2):213-227,1992.

[12] J.C.Trinkle. On the Stability and Instantaneous Velocity of Grasped Frictionless Objects. *IEEE Transactions on Robotics and Automation*, 8(5):560-572, 1992.

[13] J.C.Trinkle, J.S.Pang, S. Sudarsky, G.Lo. On Dynamic Multi-Rigid-Body Contact Problems with Coulomb Friction. To appear *Zeischrift für Angewandte Mathematik und Mechanik*.

[14] S. J. Wright. A path–following interior–point algorithm for linear and quadratic problems. Argonne National Laboratory, Mathematics and Computer Science Division. Preprint P401–1293, 1993.