4.8 The Remes Algorithm

The minimax approximation $q_n^*(x)$ to $f(x)$ can be found by using an iteration technique called the *second algorithm of Remes*. This method exploits the oscillation property of the minimax approximation as described in Theorem 4.10.

We describe one iteration step.

(S1) Given n+2 nodes

$$-1 \le x_0 < x_1 < \dots < x_{n+1} \le 1, \tag{4.8.1}$$

determine the polynomial $q(x)$ for which degree$(q) \le n$ and

$$f(x_i)-q(x_i)=(-1)^i E, \quad i=0,1,\dots,n+1. \tag{4.8.2}$$

The nodes should be so chosen that $E \ne 0$. This is a system of n+2 linear equations in which the unknowns are E and the n+1 coefficients of $q(x)$.

(S2) Having determined $q(x)$, calculate n+2 new node points

$$-1 \le z_0 < z_1 < \dots < z_{n+1} \le 1. \tag{4.8.3}$$

for which $f(x)-q(x)$ is a local optimum at each z_i, and for which $f(x_i)-q(x_i)$ and $f(z_i)-q(z_i)$ have the same sign, $i=0,1,\dots,n+1$. The first requirement implies

$$f'(z_i)-q'(z_i)=0, \quad i=1,2,\dots,n, \tag{4.8.4}$$

and also at z_0 and z_{n+1} if they are in $(-1,1)$. Also, the nodes z_i should be chosen so that

$$\|f-q\|_\infty = |f(z_k)-q(z_k)| \tag{4.8.5}$$

for some z_k.

(S3) Using Theorem 4.9 and the properties of the nodes $\{z_i\}$, we have

$$m=\operatorname*{Min}_i |f(z_i)-q(z_i)| \le \rho_n(f) \le M \equiv \operatorname*{Max}_i |f(z_i)-q(z_i)|. \tag{4.8.6}$$

If M/m is sufficiently close to 1, we consider that $q(x)$ is close enough to the minimax for practical purposes; for example, if

$$M/m \le 1.05. \tag{4.8.7}$$

If this is not true, then we set the nodes $\{x_i\}$ equal to $\{z_i\}$, and return to (S1).

The initial guess of nodes $\{x_i\}$ in (4.8.1) are usually the points

$$x_i = \cos\left[\frac{i\pi}{n+1}\right], \quad i=0,1,\ldots,n+1,$$

discussed in the last section. The first approximation $q(x)$ is therefore the polynomial $F_n(x)$ of §4.7, satisfying (4.7.31), (4.7.38), and (4.7.39). The convergence to $q_n^*(x)$ is quite rapid. It is shown in Meinardus (1967, Theorem 84, p. 111) that subject to mild restrictions on $f(x)$, we have

$$\left|\rho_n(f)-\|f-q^{(j+1)}\|_\infty\right| \le c\left|\rho_n(f)-\|f-q^{(j)}\|_\infty\right|^2, \quad j\ge 1,$$

for some $c>0$. The notation $q^{(j)}(x)$ denotes the $q(x)$ obtained in the j^{th} iteration of the Remes algorithm. A similar quadratic convergence result also holds for $\|q_n^*-q^{(j)}\|_\infty$.

Example We show the computation of $q_2^*(x)$ for $f(x)=e^x$ on $[-1,1]$ by the Remes algorithm.

We begin with

$$x_0=-1.0, \quad x_1=-.5, \quad x_2=.5, \quad x_3=1.0,$$

and we write $q(x)$ in the standard form

$$q(x)=a_0+a_1x+a_2x^2.$$

The iterates $q^{(1)}(x)$ and $q^{(2)}(x)$ are summarized in Tables 4.11 and 4.12.

Table 4.11 Remes iterate $q^{(1)}(x)$ for $q_2^*(x) \doteq e^x$

a_i, E	z_i	$f(z_i)-q(z_i)$
.989141	-1.0	-.0443369
1.130864	- .438621	.0452334
.553940	.560939	-.0454683
E=.0443369	1.0	.0443369

For this first iterate, the ratio M/m=1.026; and if the test (4.8.7) was being used, the first iterate would be adequate. But for purposes of illustration, we give the second iterate.

Table 4.12 Remes iterate $q^{(2)}(x)$ for $q_2^*(x) \doteq e^x$

a_i, E	z_i	$f(z_i)-q(z_i)$
.989039	-1.0	-.0450171
1.130184	- .436958	.0450177
.554041	.560059	-.0450174
E=.0450171	1.0	.0450171

The third iterate agrees with the second one to the number of places shown, and the values of the error are all

$$f(z_i)-q(z_i)=\pm.045017388403.$$

This is far more accuracy than is needed, based on the limited accuracy of $\rho_2(f) \doteq .045$ in the minimax approximation. But it illustrates the rapid convergence of the Remes algorithm.

There are other variants of the Remes algorithm for calculating $q_n^*(x)$. For a discussion of them, see Meinardus (1967, Chap. 7) and Powell (1981, Chap. 8-10).