3.8 Multivariate Interpolation

Interpolation of functions of more than one variable has become increasingly important in the past few years. For example, multivariate interpolation is central to the finite element method for solving partial differential equations and to three dimensional computer graphics. In this section we will introduce a few basic ideas about multivariate interpolation, while restricting ourselves to functions $f(x,y)$ of two variables. We begin with linear interpolation, in order to illustrate that multivariable interpolation differs in some significant ways from univariate [single variable] interpolation.

Let three points (x_1,y_1), (x_2,y_2), and (x_3,y_3) be given, and let data values z_1, z_2, z_3 be given at these points. Consider finding a linear polynomial

$$p(x,y)=a+bx+cy \tag{3.8.1}$$

for which

$$p(x_i,y_i) = z_i, \quad i=1,2,3 \tag{3.8.2}$$

We chose three interpolation points (x_i,y_i) because a linear polynomial $p(x,y)$ has three degrees of freedom. This illustrates that the number of interpolation nodes (x_i,y_i) is a function of the degrees of freedom in the interpolating function. For example, using n=2 node points is generally unsuitable for most forms of *bivariate* [two variable] interpolation.

The solution to (3.8.2) is given by

$$p(x,y) = z_1\ell_1(x,y)+z_2\ell_2(x,y)+z_3\ell_3(x,y) \tag{3.8.3}$$

with

$$\ell_1(x,y) = \frac{1}{d}\mathrm{Det}\begin{bmatrix} 1 & x & y \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}, \qquad \ell_2(x,y) = \frac{1}{d}\mathrm{Det}\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x & y \\ 1 & x_3 & y_3 \end{bmatrix}$$

$$\ell_3(x,y) = \frac{1}{d}\mathrm{Det}\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x & y \end{bmatrix} \qquad (3.8.4)$$

provided

$$d = \mathrm{Det}\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \neq 0 \qquad (3.8.5)$$

This last condition can be shown to be equivalent to requiring that the points (x_1,y_1), (x_2,y_2), (x_3,y_3) not be collinear. Thus distinctness of the interpolation nodes is not sufficient to yield unique solvability of (3.8.2), unlike the situation with univariate polynomial interpolation.

The formula (3.8.3) is a generalization to multivariate interpolation of the Lagrange formula (3.1.6) for univariate polynomial interpolation. The functions ℓ_i are all linear polynomials, and they satisfy

$$\ell_i(x_j,y_j)=\delta_{ij} \qquad i,j=1,2,3 \qquad (3.8.6)$$

A discussion of the error in using linear interpolation is given later in Theorem 3.7.

<u>Product formulas</u> The easiest way to construct multivariate interpolation formulas is to base them on univariate formulas, like those given earlier in the chapter. Let

$$\sum_{i=0}^{m} f(x_i)\ell_{i,m}(x)$$

interpolate $f(x)$ at $x=x_0,\ldots,x_m$; and let

$$\sum_{j=0}^{n} g(y_j)\ell_{j,n}(y)$$

interpolate g(y) at $y=y_0,\ldots,y_n$. Then the Lagrange type formula

$$p_{m,n}(x,y)=\sum_{i=0}^{m}\sum_{j=0}^{n} f(x_i,y_j)\ell_{i,m}(x)\ell_{j,n}(y) \qquad (3.8.7)$$

interpolates f(x,y) at (x_i,y_j), $0\le i\le m$, $0\le j\le n$. The interpolation nodes form a rectangular grid, an example of which is given in Figure 3.8.

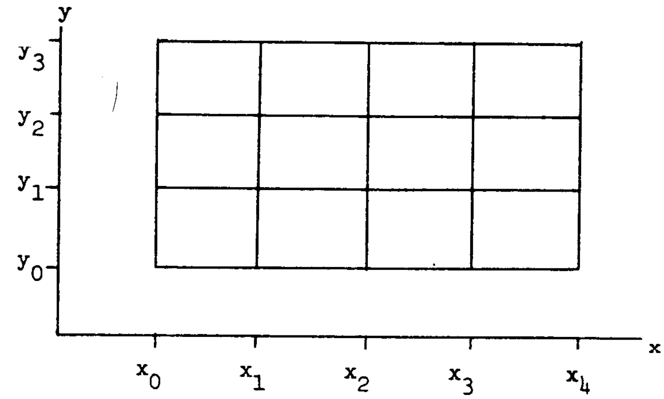

Figure 3.8 A rectangular interpolation grid.

The interpolation function $p_{m,n}(x,y)$ is a polynomial of degree$\le$m+n, although not all polynomials of degree m+n can be so represented.

<u>Example</u> Let m=n=1, and for simplicity, let $x_0=y_0=0$, $x_1=y_1=1$. Then

$$\begin{aligned} p_{1,1}(x,y) &= f(0,0)(x-1)(y-1) - f(1,0)x(y-1) \\ &\quad - f(0,1)(1-x)y + f(1,1)xy \end{aligned} \qquad (3.8.8)$$

which is of degree$\le$2. For $f(x,y)=x^2$,

$$p_{1,1}(x,y) = x \ne f(x,y)$$

Thus not all degree 2 polynomials can be represented by (3.8.8).

The formula $p_{1,1}(x,y)$ can be rewritten in the general form

$$p_{1,1}(x,y)=a+bx+cy+dxy \qquad (3.8.9)$$

Polynomials of this form are called *bilinear*. For fixed y, $p_{1,1}(x,y)$ is linear in x; and for fixed x, it is linear in y.

The following error analysis for the interpolation formula (3.8.7) is a straightforward application of the error results for univariate interpolation. Improved results by other methods are possible, but for reasons of space, we omit them.

<u>Theorem 3.6</u> Define the rectangular region

$$D = \{(x,y) \mid a \le x \le b,\ c \le y \le d\}, \qquad (3.8.10)$$

and assume the points (x_i, y_i) used in (3.8.7) are contained in D. Assume $\partial^m f(x,y)/\partial x^m$ and $\partial^n f(x,y)/\partial y^n$ exist and are continuous for all $(x,y)\in D$. Then for any $(x,y)\in D$,

$$|f(x,y)-p_{m,n}(x,y)| \le \frac{|\Psi_m(x)|}{(m+1)!} \underset{a\le\xi\le b}{\text{Maximum}} \left|\frac{\partial^m f(\xi,y)}{\partial \xi^m}\right| + \Lambda_m(x)\frac{|\Psi_n(y)|}{(n+1)!} \cdot \underset{(\xi,\eta)\in D}{\text{Maximum}} \left|\frac{\partial^n f(\xi,n)}{\partial \eta^n}\right| \qquad (3.8.11)$$

In this formula,

$$\Psi_m(x)=(x-x_0)\ldots(x-x_m), \quad \Psi_n(y)=(y-y_0)\ldots(y-y_n)$$
$$\Lambda_m(x) = \sum_{i=0}^{m} |\ell_{i,m}(x)| \qquad (3.8.12)$$

<u>Proof</u>. Write the error as

$$f(x,y)-p_{m,n}(x,y)=\left[f(x,y)-\sum_{i=0}^{m} f(x_i,y)\ell_{i,m}(x)\right]$$

$$+\sum_{i=0}^{m}\ell_{i,m}(x)\left[f(x_i,y)-\sum_{j=0}^{n} f(x_i,y_j)\ell_{j,n}(y)\right]$$

Apply the error results for univariate interpolation to each of these terms, which are univariate errors.

An analogous formula can be obtained by splitting $f(x,y)-p_{m,n}(x,y)$ with the roles of x and y reversed:

$$|f(x,y)-p_{m,n}(x,y)| \le \frac{|\psi_n(y)|}{(n+1)!}\underset{c\le\nu\le d}{\text{Maximum}}\left|\frac{\partial^n f(x,\eta)}{\partial\eta^n}\right|$$

$$+\ \Lambda_n(y)\frac{|\psi_m(x)|}{(m+1)!}\underset{(\xi,\eta)\in D}{\text{Maximum}}\left|\frac{\partial^m f(\xi,\eta)}{\partial\xi^m}\right|$$

■

<u>Example</u> Take m=n=2,

$$h_x=(b-a)/m, \quad h_y=(d-c)/n$$

$$x_i=a+ih_x, \quad y_j=c+jh_y$$

Then $\Lambda_2(x)\le 5/4$ for $a\le x\le b$. Using (3.8.11) and (3.5.5), we have that for all $(x,y)\in D$,

$$|f(x,y)-p_{2,2}(x,y)| \le \left[\frac{\sqrt{3}h_x^3}{27}\right]\underset{(\xi,\eta)\in D}{\text{Maximum}}\left|\frac{\partial^3 f(\xi,\eta)}{\partial\xi^3}\right|$$

$$+\ \left[\frac{5}{4}\right]\left[\frac{\sqrt{3}h_y^3}{27}\right]\underset{(\xi,\eta)\in D}{\text{Maximum}}\left|\frac{\partial^3 f(\xi,\eta)}{\partial\eta^3}\right|, \tag{3.8.14}$$

which is a uniform bound over D for the interpolation error.

Numerical methods for solving integral and partial differential equations often use piecewise polynomial interpolating functions. Such numerical methods divide a multidimensional region D [used in defining the equation being solved] into smaller subregions, usually rectangles or triangles. A low order polynomial interpolating function is used to approximate the unknown solution function on each such subregion. The bilinear interpolating function, used in (3.8.8), is a common choice when the subregions are rectangles; and an error formula is easily derived from Theorem 3.6. For smooth piecewise polynomial functions over rectangular grids such as that in Figure 3.8, see the discussion of tensor splines in deBoor (1978, Chap. 17).

<u>Interpolation over triangles</u> At first thought, triangular regions would seem to have such a variety of shapes that it would be cumbersome to develop a theory for interpolation of functions defined on such regions. This is, however, not the case. It is possible to restrict our interest to developing interpolation formulas over one special triangular region σ, and to then transfer these formulas to any other triangular region by a simple change of variables.

Define σ by

$$\sigma = \{(s,t) \mid 0 \le s,t,s+t \le 1\} \tag{3.8.15}$$

For any other triangle Δ, let its vertices be denoted by $v_i=(x_i,y_i)$, $i=1,2,3$; see Figure 3.10. Define a function T from σ to Δ by

$$T(s,t)=uv_1+sv_2+tv_3, \qquad (s,t)\in\sigma \tag{3.8.16}$$

with $u=1-s-t$. Then $(x,y)=T(s,t)$ gives a one-to-one correspondence between the points (s,t) of σ and the points (x,y)

of Δ. The vertices q_1, q_2, q_3 of σ correspond to the vertices v_1, v_2, v_3 of Δ; and the midpoints of the sides, q_4, q_5, q_6 in σ, correspond to the midpoints of the sides v_4, v_5, v_6 of Δ. The numbers s,t,u are called the *barycentric coordinates* of the corresponding point (x,y) in Δ. The mapping (3.8.16) is easily reversed. Simply solve for (s,t) in the linear system

$$\begin{aligned} x_1+s(x_2-x_1)+t(x_3-x_1)&=x \\ y_1+s(y_2-y_1)+t(y_3-y_1)&=y \end{aligned} \tag{3.8.17}$$

We say (s,t)=Q(x,y), the inverse function to (x,y)=T(s,t).

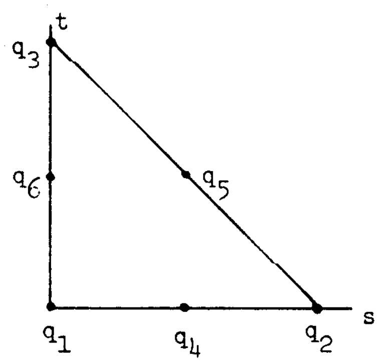

Figure 3.9 The unit simplex σ

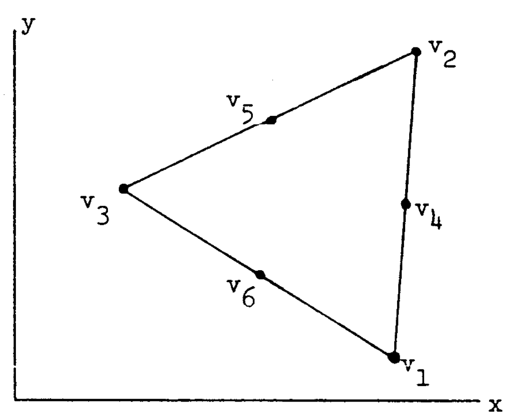

Figure 3.10 An Arbitrary triangle Δ

If a function f(x,y) is defined on Δ, then F(s,t)=f(T(s,t)) is defined on σ; and conversely, if F(s,t) is defined on σ, then

$f(x,y)=F(Q(x,y))$ is defined on Δ. An important aspect of this is that if $f(x,y)$ is a polynomial of degree r on Δ, then $F(s,t)=f(T(s,t))$ is a polynomial of degree r on σ. Thus interpolating polynomials over σ convert to interpolating polynomials of the same degree over Δ. We will give two cases of such interpolating polynomials.

Consider interpolating the function $F(s,t)$ using a linear polynomial. In particular, find a linear polynomial $p_1(s,t)$ for which

$$p_1(q_j)=F(q_j), \quad j=1,2,3, \tag{3.8.18}$$

thus interpolating $F(s,t)$ at the vertices of σ. Introduce the linear polynomials

$$\ell_1(s,t)=u=1-s-t, \qquad \ell_2(s,t)=s, \qquad \ell_3(s,t)=t \tag{3.8.19}$$

Then

$$p_1(s,t) = \sum_{j=1}^{3} F(q_j)\ell_j(s,t), \qquad (s,t)\in\sigma \tag{3.8.20}$$

is the solution to (3.8.18). To interpolate $f(x,y)$ at the vertices of Δ, use

$$P_1(x,y)= \sum_{j=1}^{3} f(v_j)\ell_j(Q(x,y)), \qquad (x,y)\in\Delta \tag{3.8.21}$$

based on (3.8.20) and the mapping $(s,t)=Q(x,y)$. This formula for $P_1(x,y)$ can be shown to be the formula (3.8.3) given earlier for linear interpolation.

For quadratic polynomial interpolation of σ, note that a general quadratic polynomial has six degrees of freedom:

$$p(s,t)=a_1+a_2s+a_3t+a_4s^2+a_5st+a_6t^2$$

Thus we should expect to impose six interpolating conditions on a quadratic interpolation polynomial. We do this by requiring

$$p_2(q_j)=F(q_j), \quad j=1,\ldots,6 \qquad (3.8.22)$$

with q_4, a_5, q_6 the midpoints of σ, as shown in Figure 3.9.

To construct $p_2(s,t)$, we proceed in analogy with (3.8.19)-(3.8.20). Find quadratic polynomials $\ell_i(s,t)$ for which

$$\ell_i(q_j)=\delta_{ij}, \qquad 1\leq i,j\leq 6. \qquad (3.8.23)$$

These are given by

$$\begin{array}{ll} \ell_1(s,t)=u(2u-1) & \ell_4(s,t)=4su \\ \ell_2(s,t)=s(2s-1) & \ell_5(s,t)=4st \\ \ell_3(s,t)=t(2t-1) & \ell_6(s,t)=4tu \end{array} \qquad (3.8.24)$$

The interpolation polynomial is given by

$$p_2(s,t)=\sum_{j=1}^{6} F(q_j)\ell_j(s,t). \qquad (3.8.25)$$

For interpolation to $f(x,y)$ on Δ, proceed in analogy with (3.8.21).

Error formulas for interpolation of $f(x,y)$ over Δ are derived by first studying the corresponding problem on σ. To simplify the presentation, we will consider only quadratic interpolation.

<u>Theorem 3.7</u> Let $f(x,y)$ be three times continuously differentiable on Δ [i.e., all partial derivatives of $f(x,y)$, of order≤ 3, are continuous on Δ]. Define

$$\delta=\text{diameter}(\Delta)=\underset{v,w\in\Delta}{\text{Maximum}}|v-w|$$

with $|v-w|$ the distance between v and w. Then for the quadratic polynomial $P_2(x,y)\equiv p_2(Q(x,y))$ that interpolates $f(x,y)$ at the nodes $v_1,\ldots,v_6$,

$$\underset{(x,y)\in\Delta}{\text{Maximum}} |f(x,y)-P_2(x,y)| \le c_2\delta^3 \cdot \underset{\substack{i+j=3\\ i,j\ge 0}}{\text{Maximum}} \left[\underset{(x,y)\in\Delta}{\text{Maximum}} \left| \frac{\partial^3 f(x,y)}{\partial x^i \partial y^j} \right| \right] \tag{3.8.26}$$

with c_2 a constant independent of (x,y) and f.

<u>Proof</u> Expand $F(s,t)=f(T(s,t))$ as a Taylor polynomial of degree 2 about $(0,0)$:

$$F(s,t)=\mathscr{P}_2(s,t) + \frac{1}{2}\int_0^1 (1-\upsilon)^2 \frac{d^3}{d\upsilon^3}[F(\upsilon s,\upsilon t)]d\upsilon. \tag{3.8.27}$$

Here $\mathscr{P}_2(s,t)$ is the quadratic Taylor polynomial, based on (1.1.12). The error formula uses the integral form of the remainder, but otherwise is the same in its derivation as that given in (1.1.12).

Write (3.8.27) as

$$F(s,t)=\mathscr{P}_2(s,t)+R_2(s,t)$$

For the error in quadratic interpolation,

$$F(s,t)-\sum_{j=1}^{6} F(q_j)\ell_j(s,t)=R_2(s,t)-\sum_{j=1}^{6} R_2(q_j)\ell_j(s,t) \tag{3.8.28}$$

because the error in interpolating $\mathscr{P}_2(s,t)$ is zero due to its being quadratic. This leads to

$$F(s,t)-p_2(s,t)= \frac{1}{2}\int_0^1 (1-\upsilon)^3 \left[\frac{d^3F(\upsilon s,\upsilon t)}{d\upsilon^3} - \sum_{j=1}^{6} \frac{d^3F(\upsilon q_j)}{d\upsilon^3}\ell_j(s,t)\right]d\upsilon \tag{3.8.29}$$

As an example of the derivatives,

$$\frac{d}{d\upsilon}F(\upsilon s,\upsilon t)=sF_1(\upsilon s,\upsilon t)+tF_2(\upsilon s,\upsilon t)$$

with the functions F_1 and F_2 denoting the derivatives of F with respect to its first and second variables, respectively.

To relate these to derivatives of $f(x,y)$, use

$$F(s,t)=f(x_1+s(x_2-x_1)+t(x_3-x_1),\ y_1+s(y_2-y_1)+t(y_3-y_1))$$

Differentiating,

$$F_1(s,t)=(x_2-x_1)f_1+(y_2-y_1)f_2$$
$$F_2(s,t)=(x_3-x_1)f_1+(y_3-y_1)f_2$$

Then for $i=1,2$,

$$\underset{(s,t)\in\sigma}{\text{Maximum}}|F_i(s,t)|$$
$$\leq [\text{Max}\{|v_2-v_1|,|v_3-v_1|\}]\cdot\underset{j=1,2}{\text{Maximum}}\left[\underset{(x,y)\in\Delta}{\text{Max}}|f_j(x,y)|\right]$$

This can be generalized to the higher order derivatives of F, and then it can be substituted into (3.8.29) to yield (3.8.26). An actual value can be calculated for the constant C_2 in (3.8.26), but generally it is unrealistically large.

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The subject of multivariate piecewise polynomial interpolation, over triangular and rectangular subregions, is growing rapidly. Of special interest in computer graphics is the construction of interpolating polynomials whose graphs join smoothly. Spline functions can be used when the grid is rectangular, but in general other methods must be used. As introductions to such questions, see Barnhill (1977) and Pavlidis (1982).