Dual counterpart intuitionistic logic

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We introduce dual counterpart intuitionistic logic (or **DCInt**): a constructive logic that is a conservative extension of intuitionistic logic, a sublogic of bi-intuitionistic logic, has the logical duality property of classical logic, and also retains the modal character of its interpretation of the connective dual to intuitionistic implication. We define its Kripke semantics along with the corresponding notion of a bisimulation, and then prove that it has both the disjunction property and (its dual) the constructible falsity property. Also, for any class $\mathcal{C}$ of Kripke frames from our semantics, we identify a condition such that $\mathcal{C}$ will have the disjunction property if it satisfies the condition. This provides a method for generating extensions of **DCInt** that retain the disjunction property.

1 Introduction

Intuitionistic logic (or **Int**) is the preeminent constructive logic, and its prime status in this category is related to the fact that it exhibits the *disjunction property*. In [27], Rauszer introduced bi-intuitionistic logic (or **BiInt**) with the intention of creating a variant of **Int** with a duality property that is analogous to that of classical logic (or **CL**).[1] In the introduction she writes

> From those investigations it appeared that an intuitionistic logic with two negations and two implications, dual to itself, would have a more elegant algebraic and model-theoretic theory than an ordinary intuitionistic logic. The purpose of this paper is to develop that theory.

As she describes, **BiInt** has duality because its semantics interprets the language of **Int** extended to include a connective that is dual to the implication connective. We refer to this connective as *exclusion*, and we write $B \prec A$ to mean that the formula B *excludes* A.[2] The exclusion connective is defined in any logic that has duality because it is defined as the dual of implication, and so–for example–it is defined in **CL**. We will elaborate further upon the concepts of duality and exclusion in Section 2. **BiInt** achieves duality while remaining a conservative extension of **Int**. Even though it extends **Int** conservatively, **BiInt** fails to be constructive because it lacks the disjunction property. This eliminates **BiInt** from consideration for any purpose where constructiveness is highly valued. The failure of **BiInt** to be constructive is strangely incongruous considering the fact that it originates from **Int**, and so this motivates investigation

[1] In [13], Drobyshevich, Odintsov, and Wansing point out that Moisil independently discovered a closely related system many years prior to Rauszer's work.

[2] The same terminology is used in [32]. Exclusion is sometimes called co-implication, subtraction, or pseudo-difference (see footnote 2 of [31] for a helpful enumeration of which authors have used which terminology for exclusion). Furthermore, the nomenclature we use here differs from Rauszer's, as she used the name Heyting-Brouwer logic (or H-B logic) instead of bi-intuitionistic logic and referred to the exclusion connective as Brouwerian implication. She also extended the language with a unary connective she called Brouwerian negation, but this connective is definable in terms of Brouwerian implication (we define it in Section 3). Finally, sometimes others have also referred to her logic as subtractive logic (for example, in [6]).

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into possible alternative approaches to extending **Int** in a way that obtains duality. The only known alternative is the logic of constructible falsity (or **N**) of [24] and [23], since it is a constructive logic that has duality and is also a conservative extension of **Int**. Though **N** does have the disjunction property, it semantically interprets the exclusion connective in a manner that is more akin to the **CL** interpretation of exclusion. This is a significant departure from how **BiInt** interprets the exclusion connective, and as a result of this **N** is not a sublogic of **BiInt**.

In this article we contribute a new alternative to **BiInt** called dual counterpart intuitionistic logic (or **DCInt**). **DCInt** is a constructive logic that is a conservative extension of **Int**, a sublogic of **BiInt**, and also has duality. This new logic is important because of its differences when compared to **BiInt** and **N**. Specifically, it is distinct from these logics on the following points:

1. **DCInt** has the disjunction property, while **BiInt** does not.
2. **DCInt** semantically interprets the exclusion connective in a way that is similar to the interpretation taken by **BiInt**. As a result, it is a sublogic of **BiInt**, while **N** is not.

DCInt is unique in that it is the only known logic that combines these two features while also obtaining duality.

The meaning of Point 1 is clear, but Point 2 requires elaboration. Rauszer defined the semantics of **BiInt** by extending the Kripke semantics for **Int** to cover exclusion, and so from the point of view of their Kripke semantics this is the sole distinction between the two. The main characteristic of a Kripke semantics is that it defines the semantic relation in way that incorporates a set of *possible worlds* together with a *reachability relation* between those worlds. The truth value of a formula is then defined relative to a particular world w, and in some cases the truth value will further depend on a quantification over non-local worlds that are reachable from w. In the Kripke semantics of **Int**, the interpretation of a formula involving the implication connective universally quantifies over non-local reachable worlds. The **BiInt** interpretation of a formula involving the exclusion connective existentially quantifies over non-local reachable worlds, and this is natural because in classical logic the existential quantifier connective is dual to the universal quantifier connective. In the context of a logic, we will call a connective *modal* when the Kripke semantics of the logic interprets the connective at a world w in such a way that its truth value depends on a quantification over worlds reachable from w; and we will call a connective *non-modal* when the Kripke semantics of the logic interprets the connective at a world w in such a way that its truth value depends only on w.[3] In this sense, the exclusion connective of **BiInt** is a modal connective. This interpretation is only natural because in the Kripke semantics of **Int** the intuitionistic implication connective is also a modal connective. In contrast, **N** interprets exclusion as a non-modal connective, and this is similar to how exclusion is interpreted in the semantics of **CL**. This similarity holds because one can show that the semantics of **CL** is embedded in the Kripke semantics of **Int**, where each **CL** structure is represented by an **Int** structure that has only a single world. With only a single world the quantification involved in the interpretation of a formula involving implication becomes trivialized, and so this means that from the view of a Kripke semantics the connectives of **CL** are all non-modal. Finally, the Kripke semantics of **DCInt** interprets exclusion as a modal connective, and so it is more similar to **BiInt** than **N**. Moreover, it is similar enough that **DCInt** turns out to be a sublogic of **BiInt**, and yet it is different enough that **DCInt** retains the disjunction property.

The rest of this paper is organized as follows. In Sections 2 and 3 we define the **CL** property of duality

[3] We use this terminology because in the Kripke semantics of modal logic the interpretation of a modal formula (for example, $\Box\phi$, or $\Diamond\phi$) quantifies over non-local reachable worlds.

$$\textbf{CL formula } A, B ::= \sigma \mid \neg A \mid A \wedge B \mid A \vee B \qquad (\sigma \in \Sigma)$$

(1a) The formulas of **CL** over Σ variables

$$\begin{array}{lll} M \vDash_{\mathbf{C}} \sigma & \text{iff} & M(\sigma) = T \\ M \vDash_{\mathbf{C}} \neg A & \text{iff} & M \nvDash_{\mathbf{C}} A \\ M \vDash_{\mathbf{C}} A \wedge B & \text{iff} & M \vDash_{\mathbf{C}} A \text{ and } M \vDash_{\mathbf{C}} B \\ M \vDash_{\mathbf{C}} A \vee B & \text{iff} & M \vDash_{\mathbf{C}} A \text{ or } M \vDash_{\mathbf{C}} B \end{array}$$

(1b) Two valued semantic interpretation of **CL**

$$\begin{aligned} \delta_{\mathbf{C}}(\sigma) &= \sigma \\ \delta_{\mathbf{C}}(\neg A) &= \neg \delta_{\mathbf{C}}(A) \\ \delta_{\mathbf{C}}(A \wedge B) &= \delta_{\mathbf{C}}(B) \vee \delta_{\mathbf{C}}(A) \\ \delta_{\mathbf{C}}(A \vee B) &= \delta_{\mathbf{C}}(B) \wedge \delta_{\mathbf{C}}(A) \end{aligned}$$

(1c) The duality correspondence of **CL**

$$M \vDash_{\mathbf{C}} \sigma \text{ iff } \delta_{\mathbf{C}}(M) \nvDash_{\mathbf{C}} \sigma \tag{1}$$
$$M \vDash_{\mathbf{C}} \neg A \text{ iff } \delta_{\mathbf{C}}(M) \nvDash_{\mathbf{C}} \neg \delta_{\mathbf{C}}(A) \tag{2}$$
$$M \vDash_{\mathbf{C}} A \wedge B \text{ iff } \delta_{\mathbf{C}}(M) \nvDash_{\mathbf{C}} \delta_{\mathbf{C}}(B) \vee \delta_{\mathbf{C}}(A) \tag{3}$$
$$M \vDash_{\mathbf{C}} A \vee B \text{ iff } \delta_{\mathbf{C}}(M) \nvDash_{\mathbf{C}} \delta_{\mathbf{C}}(B) \wedge \delta_{\mathbf{C}}(A) \tag{4}$$

(1d) The duality of **CL** semantics

and also recapitulate the semantics of **CL**, **Int**, and **BiInt**. These sections include explanations for why **CL** and **BiInt** both have duality, and for why **Int** does not. In Section 4 we define the Kripke semantics of **DCInt**, and then establish some basic results from the definitions. In particular, in Section 4.3 we prove that **DCInt** has a duality property that is analogous to that of **CL**; in Section 4.4 we prove that **DCInt** is both a conservative extension of **Int** and a sublogic of **BiInt**; and in Section 4.5 we establish the definition of a bisimulation between two **DCInt** Kripke structures. In Section 5 we prove that **DCInt** has the disjunction property and the constructible falsity property. The proof utilizes the frame fusion/gluing technique from the model theory of both modal and intuitionistic logic. This technique is utilized–for example–in [33] and in Section 6.4 of [9]. However, our application of it here is a novel contribution because formulas in **DCInt** semantics are sensitive to worlds that are both forwardly and backwardly reachable, and this complicates the application of this technique. Finally, we discuss related work in Section 6 before concluding in Section 7. Notably, the proof theory of **DCInt** is not discussed here, though we plan on investigating it in future work.

Throughout this paper we will use classical logic as the metalogic of the definitions and proofs. Also, when we mention the name of a definition or result we will sometimes also mention its numerical index N using an expression of the form "{N}" (for example: we will recall the definition of the disjunction property {7} in Section 3.1, and the definition of a conservative extension {33} in Section 4.4).

2 The duality of classical logic

The duality of **CL** can be expressed as a bijective correspondence between structurally identical formulas, where **(1)** the only difference between two corresponding formulas is that each connective is exchanged for its dual; and **(2)** the semantic interpretation of a formula A is equivalent to the "dual interpretation" of the dual of A, where the "dual interpretation" is obtained by swapping the roles of true and false. Essentially the intuition for the property of duality is that each connective will have a dual, and in the context of the "dual interpretation" that dual connective will operate in "the same way" as the original

terminology		definition
standard	verify/falsify	
M models ϕ	M verifies ϕ	iff $M \vDash_{\mathbf{C}} \phi$
M countermodels ϕ	M falsifies ϕ	iff $M \nvDash_{\mathbf{C}} \phi$

(2a) **CL** definitions of relations between structures (M) and formulas (ϕ)

term T		define term from column 1 or 2 by substituting it for "T"
standard	verify/falsify	ϕ is T iff ...
valid	verify valid	... $M \vDash_{\mathbf{C}} \phi$ for every M
satisfiable	verify satisfiable	... $M \vDash_{\mathbf{C}} \phi$ for some choice of M
unsatisfiable	falsify valid	... ϕ is not satisfiable
countermodeled	falsify satisfiable	... ϕ is not valid

(2b) **CL** definitions of the semantic classification of a formula ϕ

Figure 2: **CL** semantic definitions, where in each table the third column gives the definition, and the first and second columns describe the same notions under different naming conventions: the first column uses standard terminology and the second column uses terminology from a duality perspective. Each term in the first and second column of (b) under the "term T" heading is being defined. For each term in one of those columns, its definition is the statement determined by substituting it for "T" in its rightmost column. We will also use this substitution-style presentation for Figures 6b and 10.

connective. The language of **CL** formulas is shown in Figure 1a, and it is parameterized over a set Σ of propositional variables. Its duality correspondence is shown in Figure 1c. The correspondence arises from the symmetry inherent in both the semantic structures and the semantic interpretation of **CL** {1}. Note that Definition 1 includes the non-standard terminology of "verifies", "falsifies", and "verify/falsify valid/satisfiable". These terms are useful for discussing the mentioned "dual interpretation", and we will explain them further after introducing the concept of duality in **CL** {2}.

Definition 1. *A two-valued structure (or model) of **CL** is a function $M : \Sigma \to \{F, T\}$. For any formula ϕ, the expression $M \vDash_{\mathbf{C}} \phi$ is recursively defined on ϕ in Figure 1b. The expression $M \nvDash_{\mathbf{C}} \phi$ is true whenever $M \vDash_{\mathbf{C}} \phi$ is not true. The **models/verifies**, and **countermodels/falsifies** relations are defined in Figure 2a, and the definitions of the semantic classification of a formula are given in Figure 2b.*

A **CL** structure {1} carries the information that is needed for the semantic interpretation, and is often also referred to as a "model". Throughout the rest of this paper we will define the notion of a structure for the semantic relation $\vDash$ of other logics as well. In those contexts, a structure could also be referred to as a model, and we will also write $\nvDash$ to indicate that the relation $\vDash$ does not hold (just as we write $\nvDash_{\mathbf{C}}$).

Every **CL** structure M corresponds to a dual structure $\delta_{\mathbf{C}}(M)$ given by exchanging true and false; in other words, $\delta_{\mathbf{C}}(M) = \sigma \mapsto \text{not } M(\sigma)$. A propositional variable σ is self-dual because we have $M \vDash_{\mathbf{C}} \sigma$ iff $\delta_{\mathbf{C}}(M) \nvDash_{\mathbf{C}} \sigma$. Further, for all formulas A and B, we have the equivalences shown in Figure 1d. Equivalence 2 shows why the connective $\neg$ is self-dual, and Equivalences 3 and 4 show why the connectives $\vee$ and $\wedge$ are duals to each other. In general, this means that for every formula A and structure M we have that A being modeled by M is faithfully represented by $\delta_{\mathbf{C}}(A)$ not being modeled by $\delta_{\mathbf{C}}(M)$, and vice-versa. In other words, we have Theorem 2.

$$A \to B \equiv \neg A \vee B$$
$$B \prec A \equiv B \wedge \neg A$$

(3a) **CL** implication and exclusion

$$\delta_{\mathbf{C}}(A \to B) = \delta_{\mathbf{C}}(B) \prec \delta_{\mathbf{C}}(A)$$
$$\delta_{\mathbf{C}}(B \prec A) = \delta_{\mathbf{C}}(A) \to \delta_{\mathbf{C}}(B)$$

(3b) **CL** implication duality

$$M \vDash_{\mathbf{C}} A \to B \text{ iff } M \vDash_{\mathbf{C}} \neg A \vee B \text{ iff } \delta_{\mathbf{C}}(M) \nvDash_{\mathbf{C}} \delta_{\mathbf{C}}(B) \wedge \neg \delta_{\mathbf{C}}(A) \text{ iff } \delta_{\mathbf{C}}(M) \nvDash_{\mathbf{C}} \delta_{\mathbf{C}}(B) \prec \delta_{\mathbf{C}}(A)$$

(3c) The duality between implication and exclusion in **CL**

Theorem 2. *The function $\delta_{\mathbf{C}}$ is an involution,*[4] *and for every structure M and formula ϕ we have $M \vDash_{\mathbf{C}} \phi$ iff $\delta_{\mathbf{C}}(M) \nvDash_{\mathbf{C}} \delta_{\mathbf{C}}(\phi)$.*

This relationship between a formula and its dual demonstrates that the semantic definitions of classical logic exhibit a perfect symmetry between the dual notions of *verification* and *falsification* described by Wansing in [35]. The statement $M \vDash_{\mathbf{C}} A$ can be interpreted to mean that the structure M verifies A, so that the semantic relation $\vDash_{\mathbf{C}}$ represents the "models" relationship defined by the notion of verification. Dually, in this interpretation the statement $M \nvDash_{\mathbf{C}} A$ means that the structure M falsifies A, so that the semantic relation $\nvDash_{\mathbf{C}}$ represents the "models" relationship defined by the notion of falsification. These two interpretations are sensible because we cannot have both $M \vDash_{\mathbf{C}} A$ and $M \nvDash_{\mathbf{C}} A$ for any structure M and formula A. These interpretations manifest at the formula level as well because the verify and falsify perspectives can refer to each other through the negation connective: a structure verifies $\neg A$ iff it falsifies A, and it falsifies $\neg A$ iff it verifies A. With respect to verification, the $\wedge$ connective fulfills the conjunctive role and the $\vee$ connective fulfills the disjunctive role. The same holds for falsification but with $\vee$ and $\wedge$ swapped. For example, from this perspective the de Morgan duality schema of $\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$ is valid because the $\vee$ connective is conjunctive (since it is dual to $\wedge$) in the context of the falsification interpretation: the verification of $\neg(A \vee B)$ is equivalent to the falsification of both A and B, which is equivalent to the verification of both $\neg A$ and $\neg B$. In other words, the de Morgan duality of **CL** can be viewed as a result of the duality described by Figure 1c. It is also possible to define the duality of **CL** as a property inherent to its proof theory, but in this article we will focus entirely on duality in the model theoretic context.

Just as the negation, disjunction, and conjunction connectives have duals, the exclusion connective serves as the dual of the implication connective. Since we are defining it as the dual of the implication connective, the meaning of the exclusion connective will depend on the meaning of the implication connective in the logical context. In the context of **CL** the implication connective is defined by $A \to B \equiv \neg A \vee B$, and so the duality correspondence given by $\delta_{\mathbf{C}}$ induces the definition of classical exclusion shown in Figure 3a along with the extended definition of $\delta_{\mathbf{C}}$ in Figure 3b. Under this definition, for every **CL** structure M and formulas A, B, we have the equivalence shown in Figure 3c. The $\wedge$ connective is disjunctive from the falsification perspective, and so therefore from this perspective the formula $B \wedge \neg A$ is analogous (reading from right to left) to the formula $\neg A \vee B$ (reading from left to right) from the verify perspective. That is, the falsification of $B \wedge \neg A$ can be described as "if A is falsified, then B is falsified".[5]

Figure 2b includes the non-standard classifications of a formula as being verify valid or falsify valid. These two definitions are arrived at by extending the standard "valid" classification to the dual concepts

[4] An involution is a function that is equal to its own inverse.

[5] We write the exclusion schema as $B \prec A$ instead of $A \prec B$ in order to suggest this intuition.

of verification and falsification: a formula is verify valid iff every structure verifies it; and a formula is falsify valid iff every structure falsifies it. We use these two definitions to formally state the **CL** property of duality {3}, which follows from Theorem 2.

Theorem 3 (duality of **CL**). *For any formula ϕ:*

1. *ϕ is verify valid iff $\delta_C(\phi)$ is falsify valid*
2. *ϕ is verify satisfiable iff $\delta_C(\phi)$ is falsify satisfiable*
3. (a) *if ϕ is verify valid then $\delta_C(\phi)$ is not verify satisfiable*
   (b) *if ϕ is falsify valid then $\delta_C(\phi)$ is not falsify satisfiable*

The latter two parts of Theorem 3 are directly implied by the first part because in **CL** the classifications of "verify valid" and "falsify valid" happen to be exactly equivalent to the standard classifications of "valid" and "unsatisfiable", respectively. However since we define them in terms of the notions of verification and falsification, these two classifications will no longer be equivalent to "valid" and "unsatisfiable" in the context of a logic that has a more nuanced interpretation of verification and falsification. Section 4 will introduce the semantics of **DCInt**, and this is an example of such a logic (for example, in **DCInt** the classification of a formula as falsify valid is distinct from its classification as unsatisfiable). Furthermore, Section 3.1 will explain why this aspect of **DCInt** is important.

3 The Kripke semantics of Int, DualInt, and BiInt

Int interprets the intuitionistic propositions of Figure 4a as either *proved* or *not proved*. As such, the natural intuitionistic notion of a verified proposition is the same as a proved proposition, and the notion of a falsified proposition is the same as a proposition for which no proof is known. This view of verification and falsification is notable for its imbalance: a proposition is verified by the demonstration of a proof object, whereas a falsified proposition simply indicates the absence of such an object. In [34], Wansing explains that **Int** can be criticized on this basis.

> ...intuitionistic negation has been criticized, because it does not express the idea of a direct falsification. An intuitionistically negated formula [$\neg A$] is verified at a possible world (alias state) s in an intuitionistic Kripke model iff at every state related to s by the pre-order of the model, A fails to be verified. There is no way of falsifying A at s in the sense of verifying the negation of A by considering just s.

In addition to this criticism, there is also the problem that under this interpretation of intuitionistic verification and falsification, **Int** cannot have a duality property that is analogous to that of **CL**. This is because exclusion cannot be defined in terms of the other **Int** connectives, and it follows from Crolard's Corollary 2.19 of [6]. We will also provide an alternative proof later in this section, because it more clearly illustrates the problem that arises from seeking a duality property in the context of **Int**. As mentioned in the introduction, Rauszer introduced **BiInt** for the express purpose of developing a conservative extension of **Int** that has the duality property with respect to the intuitionistic notions of verification and falsification. It is defined on the **BiInt** propositions of Figure 4a, and its duality correspondence is shown in Figure 4b. **BiInt** has duality with respect to that correspondence because its semantic interpretation satisfies a collection of equivalences that is analogous to that of **CL** (shown in Figure 1d). The dual-intuitionistic logic (or **DualInt**) of [8] and [17] defines exclusion but not implication, and in [4] Brunner & Carnielli show that it is (as a logic) the dual of **Int**. **DualInt** is closely related to **BiInt**, and Section 4.4 proves some results about its relation to **DCInt**.

$$
\begin{array}{rll}
\textbf{Int}\ \text{formula}\ A,B & ::=\ \sigma \mid \bot \mid \top \mid A\wedge B \mid A\vee B \mid A\to B & (\sigma\in\Sigma)\\
\textbf{DualInt}\ \text{formula}\ A,B & ::=\ \sigma \mid \bot \mid \top \mid A\wedge B \mid A\vee B \mid B\prec A & (\sigma\in\Sigma)\\
\textbf{BiInt}\ \text{formula}\ A,B & ::=\ \sigma \mid \bot \mid \top \mid A\wedge B \mid A\vee B \mid A\to B \mid B\prec A & (\sigma\in\Sigma)
\end{array}
$$

(4a) The formulas of both **Int** and **BiInt** over Σ variables

$$
\begin{array}{lll}
\delta_{\mathbf{B}}(\sigma)=\sigma & & \\
\delta_{\mathbf{B}}(\top)=\bot & \delta_{\mathbf{B}}(A\to B)=\delta_{\mathbf{B}}(B)\prec\delta_{\mathbf{B}}(A) & \delta_{\mathbf{B}}(A\wedge B)=\delta_{\mathbf{B}}(A)\vee\delta_{\mathbf{B}}(B)\\
 & \delta_{\mathbf{B}}(B\prec A)=\delta_{\mathbf{B}}(A)\to\delta_{\mathbf{B}}(B) & \delta_{\mathbf{B}}(A\vee B)=\delta_{\mathbf{B}}(A)\wedge\delta_{\mathbf{B}}(B)\\
\delta_{\mathbf{B}}(\bot)=\top & &
\end{array}
$$

(4b) The duality correspondence of **BiInt**

The semantics of **Int**, **DualInt**, and **BiInt** can be given by three different but related Kripke semantic interpretations. All of them involve the same structures {5}; i.e. an **Int** structure is also a **DualInt** and **BiInt** structure. Note that in Definition 5 we write $\mathcal{P}(W)$ to denote the power set of a set W.

Definition 4 (**Int** Kripke frame)**.** *A tuple $F=\langle W,\le\rangle$ is an **Int** Kripke frame iff W is a non-empty set, and $\le$ is a preorder*[6] *on the set W. An element of W is called a world. The relation $\le$ is called the reachability relation, and we say that a world x* reaches *a world y iff $x\le y$.*

Definition 5 (**Int** Kripke structure)**.** *A tuple $M=\langle W,\le,v\rangle$ is an **Int** Kripke structure (or model) iff $\langle W,\le\rangle$ is an **Int** frame and v is a function of type $\Sigma\to\mathcal{P}(W)$, where the function v satisfies the following property: for any $w,w'\in W$, if $w\le w'$ and $w\in v(\sigma)$ then $w'\in v(\sigma)$. We define $w\in M$ to be true iff $w\in W$. The function v is called the valuation function of M.*

Definition 6 specifies how a Kripke structure M interprets a formula in the context of **Int**, **DualInt**, and **BiInt**. The definition is parameterized over a specifier that determines which of **Int**, **DualInt**, and **BiInt** is the logic under consideration.

Definition 6. *Let $(\mathbf{L},X)$ be either $(\mathbf{Int},\mathbf{I})$, $(\mathbf{DualInt},\mathbf{U})$, or $(\mathbf{BiInt},\mathbf{B})$. Let $M=\langle W,\le,v\rangle$ be an **Int** structure {5}, ϕ be an $\mathbf{L}$ formula, and $w\in W$. According to $\mathbf{L}$ and X, Figure 5 defines the relation $M,w\vDash_X\phi$ recursively on ϕ. The expression $M\vDash_X\phi$ is true iff for every $w\in W$ we have $M,w\vDash_X\phi$. Finally, Figure 6 defines several semantic definitions with respect to $\mathbf{L}$ and $\vDash_X$.*

For any semantic relation $\vDash$ of a Kripke semantics, when the structure M is implicitly clear from the context we may write $w\vDash\phi$ instead of $M,w\vDash\phi$.

We define the intuitionistic negation connective by $\neg A\equiv A\to\bot$, and we define the dual-intuitionistic negation (also referred to as "weak negation") connective by $\sim A\equiv\top\prec A$. The latter connective is sometimes pronounced "non", so that the formula $\sim A$ would be pronounced "non A". These two connectives are dual to each other because via these definitions we have $\delta_{\mathbf{B}}(\neg A)=\sim\delta_{\mathbf{B}}(A)$ and $\delta_{\mathbf{B}}(\sim A)=\neg\delta_{\mathbf{B}}(A)$. Finally, note that this definition of the dual negation connective also applies in **CL**, since exclusion is defined in **CL** and we can define $\top$ as $\sigma\vee\neg\sigma$ for some σ. In that setting dual negation is defined as $\sim A\equiv\top\wedge\neg A$, and so it is equivalent to the classical negation connective. This is perfectly appropriate, since the classical negation connective is self-dual.

[6] A binary relation is a preorder iff it is reflexive and transitive.

$$\begin{array}{lll} M,w \vDash_X \top & & \\ M,w \nvDash_X \bot & & \\ M,w \vDash_X A \wedge B & \text{iff} & M,w \vDash_X A \text{ and } M,w \vDash_X B \\ M,w \vDash_X A \vee B & \text{iff} & M,w \vDash_X A \text{ or } M,w \vDash_X B \end{array}$$

(5a) Common cases for $X \in \{\mathbf{I},\mathbf{B},\mathbf{U}\}$

$$\begin{array}{lll} M,w \vDash_\mathbf{I} \sigma & \text{iff} & w \in v(\sigma) \\ M,w \vDash_\mathbf{I} A \to B & \text{iff} & \text{for every } w' \geq w: \\ & & M,w' \nvDash_\mathbf{I} A \text{ or } M,w' \vDash_\mathbf{I} B \end{array}$$

(5b) **Int**

$$\begin{array}{lll} M,w \vDash_\mathbf{U} \sigma & \text{iff} & w \notin v(\sigma) \\ M,w \vDash_\mathbf{U} B \prec A & \text{iff} & \text{there exists } w' \geq w: \\ & & M,w' \vDash_\mathbf{U} B \text{ and } M,w' \nvDash_\mathbf{U} A \end{array}$$

(5c) **DualInt**

$$\begin{array}{lll} M,w \vDash_\mathbf{B} \sigma & \text{iff} & w \in v(\sigma) \\ M,w \vDash_\mathbf{B} A \to B & \text{iff} & \text{for every } w' \geq w: M,w' \nvDash_\mathbf{B} A \text{ or } M,w' \vDash_\mathbf{B} B \\ M,w \vDash_\mathbf{B} B \prec A & \text{iff} & \text{there exists } w' \leq w: M,w' \vDash_\mathbf{B} B \text{ and } M,w' \nvDash_\mathbf{B} A \end{array}$$

(5d) **BiInt**

Figure 5: The Kripke semantic interpretation for **Int**, **DualInt**, and **BiInt**

The **BiInt** semantic interpretation of Figure 5 is identical to the **Int** interpretation when restricted to the intuitionistic formulas, and it is almost identical to the **DualInt** interpretation when restricted to the dual-intuitionistic formulas. In the exclusion case, both **BiInt** and **DualInt** interpret the formula by existentially quantifying over non-local possible worlds. On these formulas, the main difference between the **BiInt** and **DualInt** interpretations is the direction of world reachability. **BiInt** interprets exclusion in terms of backwards reachability, while **DualInt** interprets exclusion in terms of forward reachability. These definitions are natural in terms of dualizing implication because of the definition of intuitionistic implication in Figure 5b. The interpretation of an implication formula universally quantifies over non-local possible worlds, and in first order classical logic the existential and universal connectives are dual to each other. In summary: in the interpretations in Figure 5 both implication and exclusion are modal connectives, and every other connective is non-modal.

3.1 Int does not have duality

One property that the semantic definitions of **Int**, **DualInt**, and **BiInt** share is that the truth value of a formula at a world is total, so that the world either verifies or falsifies the formula. That is, for $X \in \{\mathbf{I},\mathbf{B},\mathbf{U}\}$, structure M, world w of M, and formula ϕ, we have either $M,w \vDash_X \phi$ or $M,w \nvDash_X \phi$. This means that the notion of falsifying a formula is identical to the notion of failing to verify a formula, and vice-versa (verifying a formula ≡ failing to falsify a formula). The identification of these notions extends to validity and unsatisfiability as well, which means that the **Int** notion of a falsify valid formula is equivalent to the **Int** notion of an unsatisfiable formula. It turns out that the identification of these two notions prevents **Int** from being able to have duality, as the proof of Theorem 8 shows that **Int** having duality contradicts **Int** having the disjunction property {7}.

Definition 7 (disjunction property)**.** *A semantics of a logic has the disjunction property iff for every valid formula $A \vee B$, either A is valid or B is valid.*

Theorem 8 (**Int** does not have duality)**.** ***Int*** *cannot have duality with respect to the correspondence* $\delta_\mathbf{B}$

terminology		definition
standard	verify/falsify	
M models ϕ at w	M verifies ϕ at w	iff $M,w \vDash_X \phi$
M countermodels ϕ at w	M falsifies ϕ at w	iff $M,w \nvDash_X \phi$

(6a) Definitions of relations between structures (M), worlds (w), and formulas (ϕ), in the context of **L**.

term T		define term from column 1 or 2 by substituting it for "T"
standard	verify/falsify	ϕ is T in **L** iff ...
valid	verify valid	... $M \vDash_X \phi$ for every M
satisfiable	verify satisfiable	... $M,w \vDash_X \phi$ for some choice of M with $w \in M$
unsatisfiable	falsify valid	... ϕ is not satisfiable in **L**
countermodeled	falsify satisfiable	... ϕ is not valid in **L**

(6b) Definitions of the semantic classification of a formula ϕ.

Figure 6: Semantic definitions where $\vDash_X$ is the semantic relation for a logic **L**, and **L** is either **Int**, **DualInt**, or **BiInt**. In each table the first column is standard terminology and the second column is terminology from a duality perspective. Each term in the first and second column of (b) is defined by substituting it for "T" in the rightmost column.

and the notions of verification and falsification from Definition 6.

Proof. Suppose that the exclusion connective $\prec$ is somehow defined from the intuitionistic connectives, and suppose for sake of contradiction that **Int** has duality with respect to $\delta_{\mathbf{B}}$ of Figure 4b. This means the dual negation connective $\sim$ is also defined in terms of the intuitionistic connectives, and that for any formula ϕ, we have that ϕ is verify valid iff $\delta_{\mathbf{B}}(\phi)$ is falsify valid.

Fix a propositional variable σ. The formula $\sigma \wedge \neg\sigma$ is unsatisfiable. This is the same as being falsify valid, and so by duality we must have that its dual $\delta_{\mathbf{B}}(\sigma \wedge \neg\sigma) = \sigma \vee {\sim}\sigma$ is verify valid (i.e. valid). By the disjunction property of **Int** we must have that either ${\sim}\sigma$ or σ is valid. Clearly σ is not valid, so we must have that ${\sim}\sigma$ is valid (i.e. verify valid). By duality we must have that its dual $\neg\sigma$ is falsify valid. This means that $\neg\sigma$ is unsatisfiable, but that is a contradiction because $\neg\sigma$ is clearly satisfiable. □

The proof of Theorem 8 is congruent with the fact that **BiInt** lacks the disjunction property, since the formula $\sigma \vee {\sim}\sigma$ is one example of why **BiInt** fails to have it: the formula $\sigma \vee {\sim}\sigma$ is valid in **BiInt**, but neither σ nor ${\sim}\sigma$ is valid. The argument used in the proof will apply to any conservative extension of **Int** that has certain properties. Besides relying on the basic facts about **Int** formulas, it only needs to rely on the fact that the notion of unsatisfiable is the same as falsify valid, and that the logic has both duality and the disjunction property. This means that for a logic to be a conservative extension of **Int** and have both duality and the disjunction property, that logic must make a meaningful distinction between the notions of unsatisfiable and falsify valid. In the next section we will define the **DCInt** semantic interpretation, which is defined in such a way that it does indeed distinguish between these two notions. Unlike in the semantics of **Int**, **DualInt**, and **BiInt**, a world of a **DCInt** structure may only partially interpret a given formula, so that the formula may not have any truth value at all. From another perspective, it assigns the formula one of three truth values: verified, falsified, or *unknown* (neither verified nor falsified). **DCInt** is

$$\text{Polarity } p \ ::= \ + \mid -$$
$$\text{Formula } F \ ::= \ \sigma \mid \top_p \mid F_1 \wedge_p F_2 \mid F_1 \to_p F_2 \qquad (\sigma \in \Sigma)$$

(7a) **DCInt** formulas over Σ variables

$$\overline{+} = -$$
$$\overline{-} = +$$

(7b) Inverse of a polarity

$$\begin{aligned} \tau(\sigma) &= \sigma & \tau(A \wedge B) &= \tau(A) \wedge_+ \tau(B) \\ \tau(\bot) &= \top_- & \tau(A \vee B) &= \tau(A) \wedge_- \tau(B) \\ \tau(\top) &= \top_+ & \tau(A \to B) &= \tau(A) \to_+ \tau(B) \\ & & \tau(B \prec A) &= \tau(A) \to_- \tau(B) \end{aligned}$$

(7c) Translation from **BiInt** formulas to **DCInt** formulas

similar to **N** in this respect, as the Kripke semantic interpretations of **N** in [30] and [20] also assign one of those three truth values to a given formula.[7]

4 The Kripke semantics of DCInt

In this section we define **DCInt** via its Kripke semantics, which bears some resemblance to the semantics of both **BiInt** and **N** in [20]. To emphasize the duality inherent to **DCInt**, we define both the language of **DCInt** formulas and the semantics of **DCInt** in a polarized form that follows the approach of articles such as [34], [35], [36], and [12].

Definition 9 (**DCInt** language)**.** *Let the set* Σ *be a countable set of propositional variables. Let* $\mathbb{F}(\Sigma)$ *denote the set of formulas over* Σ *variables, which are defined in Figure 7a. In a context where the set of propositional variables is not explicitly named, the expression* Σ *will refer to that set. In a context where the expression* σ *represents a formula, the reader should assume that* σ *is a propositional variable (i.e.* $\sigma \in \Sigma$*). Figure 7a defines polarities and Figure 7b defines the inversion function* $\overline{\cdot}$ *on polarities.*

A polarity p appears in a logical connective as a syntactic parameter that selects one of two duals. A connective with a positive polarity functions in the typical way, and a connective with a negative polarity functions as its dual. For example, Figure 7c defines a bijective translation τ from the language of **BiInt** to the language of **DCInt**, and it shows that the connective $\wedge_+$ corresponds to conjunction and the connective $\to_-$ corresponds to exclusion. We define the polarized syntax of intuitionistic and dual-intuitionistic negation by $\urcorner_p A \equiv A \to_p \top_{\overline{p}}$. Naturally, the translation maps $\neg A$ to $\urcorner_+ \tau(A)$ and maps $\sim A$ to $\urcorner_- \tau(A)$. The polarized language of formulas is of course not fundamental to **DCInt**, and so it is equivalently defined on the **BiInt** formulas of Figure 4a via the translation. One benefit of the polarized style of the definitions is that it allows for a more compact presentation and reduces the length of some of the proofs (in many of the proofs, it is especially useful to be able to quantify over a polarity parameter).

In addition to the world reachability concept from the semantics of **Int**, the Kripke semantics for **DCInt** also requires the dual notion of *backwardly reaches*: a world w backwardly reaches a world v iff vRw. Definition 10 polarizes the notion of reachability so that wRv (i.e. forward reachability) is equivalent

[7] In [14], Dunn refers to the partiality of a valuation as a truth value "gap". In Section 6 he discusses the aspect of partiality that is present in the Kripke semantics of **N**.

to $(w \prec_{+} v) \in R$, and vRw (i.e. backward reachability) is equivalent to $(w \prec_{-} v) \in R$. This allows for the parameterization of the reachability direction by a polarity p (such as the statement "$(w \prec_{p} v) \in R$").

Definition 10. *Let R be a preorder relation on the set W. For every $w_0, w_1 \in W$:*

- *the expression $w_0 \prec_{+} w_1$ denotes (w_0, w_1)*
- *the expression $w_0 \prec_{-} w_1$ denotes (w_1, w_0)*

For any $W' \subseteq W$, and any polarity p, the expression $\uparrow_p(R, W')$ is called the (p)-upset of W' in R, and it denotes the set $\{w_1 \in W \mid (w_0 \prec_p w_1) \in R$ for some $w_0 \in W'\}$.

The notation of Definition 10 can be difficult to read, and so we will make frequent use of abbreviations to reduce the burden. When the relation R is distinguished within the context we simply write $\uparrow_p W'$ instead of $\uparrow_p(R, W')$. For a singleton set $\{w\} \subseteq W$ we abbreviate $\uparrow_p(R, \{w\})$ and $\uparrow_p\{w\}$ by the expressions $\uparrow_p(R, w)$ and $\uparrow_p w$, respectively. Furthermore, when the relation is clear from context, for polarity p and elements $w, w' \in W$ we may also write just $w_0 \prec_p w_1$ instead of $(w_0 \prec_p w_1) \in R$. The polar reachability lemma $\{11\}$ states some basic facts about these notational definitions, and makes use of the abbreviations wherever possible.

Lemma 11 (polar reachability)**.** *Let $\le$ be a preorder on W. For every polarity p and w_0, w_1, $w_2 \in W$:*

1. ***(a)** $w_0 \prec_p w_1$ iff $w_1 \prec_{\overline{p}} w_0$; and **(b)** $(w_0 \prec_p w_1) \in \le$ iff $(w_0 \prec_{\overline{p}} w_1) \in \le^{-1}$*
2. ***(a)** $w_1 \in \uparrow_p w_0$ iff $w_0 \in \uparrow_{\overline{p}} w_1$; and **(b)** $w_1 \in \uparrow_p(\le, w_0)$ iff $w_1 \in \uparrow_{\overline{p}}(\le^{-1}, w_0)$*
3. *if $w_0 \prec_p w_1$ and $w_1 \prec_p w_2$ then $w_0 \prec_p w_2$*
4. *if $w_1 \in \uparrow_p w_0$ and $w_2 \in \uparrow_p w_1$ then $w_2 \in \uparrow_p w_0$*
5. *if $w_1 \notin \uparrow_p w_0$ then $\uparrow_{\overline{p}} w_1 \cap \uparrow_p w_0 = \varnothing$*

Proof. The proof follows directly from checking the definitions. □

Definition 12 specifies the **DCInt** notion of a frame, which is just like a standard **Int** frame except that it also possesses a pair of functions r^+ and r^-.

Definition 12 (**DCInt** Frame)**.** *A tuple $F = \langle W, \le, r^+, r^- \rangle$ is a **DCInt** frame iff **(1)** W is a non-empty set; and **(2)** $\le$ is a preorder on the set W; and **(3)** for each polarity p, r^p is a function of type $W \to W$ such that for every $w \in W$ we have $r^p(w) \in \uparrow_{\overline{p}}(\le, w)$. The elements of W are called the worlds of F, the relation $\le$ is called the reachability relation of F, and the functions r^+ and r^- are called the positive and negative dual counterpart functions.*

When the frame F is distinguished within the context, we will abbreviate $\uparrow_p(\le, w)$ by $\uparrow_p w$. The reader may find it helpful to consider the following alternate way of stating Part 12.3: for every $w \in W$ we have $r^+(w) \le w \le r^-(w)$. The dual counterpart functions affect the interpretation of exclusion from the verification perspective and implication from the falsification perspective, and they are an essential aspect of **DCInt** semantics. We will discuss these functions in more detail later on and so for now we just state Lemma 13, which shows that it is sufficient to choose the identity function for each of these functions in order to induce a **DCInt** frame from a preorder.

Lemma 13. *For every preorder $\le$ on a non-empty set W, the tuple $\langle W, \le, \imath_W, \imath_W \rangle$ is a **DCInt** frame, where $\imath_W$ is the identity function on W.*

Proof. The identity function is acceptable as a dual counterpart function because a world is always a member of its (p)-upset, for any polarity p. □

A **DCInt** structure {14} combines a frame with a pair of dual valuations: the negative valuation v^- and the positive valuation v^+. Each valuation is just like the valuation from a standard **Int** structure, except that the persistence property is oriented in the direction that Definition 10 associates to the polarity of the valuation (i.e. negative=backward and positive=forward). The positive valuation determines which propositional variables are verified and the negative valuation determines which are falsified, so we also require that no propositional variable is both verified and falsified.

Definition 14 (**DCInt** structure)**.** *A tuple $M = \langle F, v^+, v^- \rangle$ is a **DCInt** structure (or model) over Σ variables iff F is a **DCInt** frame, and the pair of v^+ and v^- satisfies the following requirements. Each of v^+ and v^- must be a function that maps each element of Σ to a subset of the worlds of F, and must satisfy:*

1. *(valuation polar persistence) for each $\sigma \in \Sigma$ and polarity p we have $\uparrow_p v^p(\sigma) \subseteq v^p(\sigma)$.*
2. *(valuation polar consistence) for each $\sigma \in \Sigma$ we have $v^+(\sigma) \cap v^-(\sigma) = \varnothing$.*

*For a class $\mathcal{C}$ of **DCInt** frames, we will say that M is a structure of class $\mathcal{C}$ iff M has a frame that is in $\mathcal{C}$.*

For convenience, when discussing a frame $F = \langle W, \leq, r^+, r^- \rangle$ we will write $w \in F$ for $w \in W$, and when discussing a structure $M = \langle F, v^+, v^- \rangle$ we will write $w \in M$ for $w \in F$. Notice that the valuation polar consistence property allows for the possibility that a pair of valuations neither verifies nor falsifies a given propositional variable at a particular world.

The **DCInt** semantics definition {15} specifies two dual ways to interpret a **DCInt** formula with respect to a particular world of a structure, where $M, w \vDash^+ \phi$ means that ϕ is verified (or positively modeled), and $M, w \vDash^- \phi$ means that ϕ is falsified (or negatively modeled). As a consequence, for each world w of a structure M there are two distinguished sets of formulas: the verified formulas, and the falsified formulas. These two sets are denoted by the expressions $\mathbb{T}(M, w, +)$ and $\mathbb{T}(M, w, -)$, respectively. We also define the expression $M, w \vDash^p_{\mathbf{D}} \phi$ for **DCInt** on **BiInt** formulas {16} because we consider the languages of **DCInt** and **BiInt** to be essentially equivalent. We advise the reader to study Definitions 15 and 16 together, since they represent two perspectives of the same semantics.

Definition 15 (**DCInt** semantics)**.** *Let M be a **DCInt** structure over Σ variables, p be a polarity, $w \in M$, and $\phi \in \mathbb{F}(\Sigma)$. The relation $M, w \vDash^p \phi$ is recursively defined by Figure 8a. The expression $M \vDash^p \phi$ is true iff for all $u \in W$ we have $M, u \vDash^p \phi$. The expression $\mathbb{T}(M, w, p)$ denotes the set $\{\phi \in \mathbb{F}(\Sigma) \mid M, w \vDash^p \phi\}$.*

Figure 8b displays the meaning of the expression $M, w \vDash^p_{\mathbf{D}} \phi$ of Definition 16 for the cases of implication, exclusion, conjunction, and disjunction. We will typically use **DCInt** formulas when discussing general statements about **DCInt**, and use **BiInt** formulas when considering concrete examples and specific observations.

Definition 16 (**DCInt** semantics on **BiInt** formulas)**.** *For every **DCInt** structure M, polarity p, $w \in M$, and **BiInt** formula ϕ: the expression $M, w \vDash^p_{\mathbf{D}} \phi$ denotes $M, w \vDash^p \tau(\phi)$.*

The left half of Figure 8b shows that the **DCInt** interpretation is similar to the **BiInt** interpretation of Figures 5a and 5d. The right half of the Figure 8b is similar to the falsifier interpretation of **BiInt** given by the $\nvDash_{\mathbf{B}}$ relation in Table 6a. The **DCInt** interpretation differs from that of **BiInt** on two primary points: **(1)** it is partial (in the sense described in Section 3.1), while the **BiInt** interpretation is total; and **(2)** it

$M,w \vDash^p \sigma$	iff	$w \in v^p(\sigma)$
$M,w \vDash^p \top_p$		
$M,w \nvDash^p \top_{\overline{p}}$		
$M,w \vDash^p A \to_p B$	iff	for every $w' \in \uparrow_p w$, $M,w' \nvDash^p A$ or $M,w' \vDash^p B$
$M,w \vDash^p A \to_{\overline{p}} B$	iff	there exists $w' \in \uparrow_{\overline{p}} w$, $M, r^{\overline{p}}(w') \vDash^{\overline{p}} A$ and $M,w' \vDash^p B$
$M,w \vDash^p A \wedge_p B$	iff	$M,w \vDash^p A$ and $M,w \vDash^p B$
$M,w \vDash^p A \wedge_{\overline{p}} B$	iff	$M,w \vDash^p A$ or $M,w \vDash^p B$

(8a) Kripke semantic interpretation of **DCInt** formulas

$M,w \vDash^+_{\mathbf{D}} A \wedge B$	iff	$M,w \vDash^+_{\mathbf{D}} A$ and $M,w \vDash^+_{\mathbf{D}} B$	$M,w \vDash^-_{\mathbf{D}} A \vee B$	iff	$M,w \vDash^-_{\mathbf{D}} A$ and $M,w \vDash^-_{\mathbf{D}} B$
$M,w \vDash^+_{\mathbf{D}} A \vee B$	iff	$M,w \vDash^+_{\mathbf{D}} A$ or $M,w \vDash^+_{\mathbf{D}} B$	$M,w \vDash^-_{\mathbf{D}} A \wedge B$	iff	$M,w \vDash^-_{\mathbf{D}} A$ or $M,w \vDash^-_{\mathbf{D}} B$
$M,w \vDash^+_{\mathbf{D}} A \to B$	iff	for every $w' \geq w$, $M,w' \nvDash^+_{\mathbf{D}} A$ or $M,w' \vDash^+_{\mathbf{D}} B$	$M,w \vDash^-_{\mathbf{D}} B \prec A$	iff	for every $w' \leq w$, $M,w' \nvDash^-_{\mathbf{D}} A$ or $M,w' \vDash^-_{\mathbf{D}} B$
$M,w \vDash^+_{\mathbf{D}} B \prec A$	iff	there exists $w' \leq w$, $M,w' \vDash^+_{\mathbf{D}} B$ and $M, r^-(w') \vDash^-_{\mathbf{D}} A$	$M,w \vDash^-_{\mathbf{D}} A \to B$	iff	there exists $w' \geq w$, $M,w' \vDash^-_{\mathbf{D}} B$ and $M, r^+(w') \vDash^+_{\mathbf{D}} A$

(8b) **DCInt** interpretation of the complex cases of **BiInt** formulas

interprets falsified implication and verified exclusion using the dual counterpart functions. First we will explain the former point in Remark 17, and then we will discuss the latter point in detail.

Remark 17. *The **DCInt** interpretation of a formula ϕ is partial in the sense that a world w of a structure M may satisfy $M,w \nvDash^+ \phi$ and $M,w \nvDash^- \phi$, so that ϕ is neither verified nor falsified at that world. This aspect of the semantics is necessary to avoid the problem demonstrated by the proof of Theorem 8, since it produces a semantics in which the statements "$\sigma \wedge_+ \overline{\top}_+ \sigma$ is unsatisfiable" and "$\sigma \wedge_+ \overline{\top}_+ \sigma$ is falsify valid" have distinct meanings (these statements are formally defined for **DCInt** below, in Definition 19).*

To understand the interpretation of verified exclusion and falsified implication, it is critical to understand two important properties of **DCInt** semantics. The polar persistence and consistence properties {18} state that both Properties 1 and 2 of a **DCInt** structure {14} extend to complex formulas.

Theorem 18. *For any **DCInt** structure M, formula ϕ, world $w \in M$, and polarity p:*

- *(polar persistence property) if $M,w \vDash^p \phi$, then for every world $w' \in \uparrow_p w$ we have $M,w' \vDash^p \phi$.*
- *(polar consistence property) if $M,w \vDash^p \phi$, then $M,w \nvDash^{\overline{p}} \phi$.*

With respect to their interpretations of verification and falsification (see Figure 6), an analogous form of this theorem holds for **Int**, **DualInt**, and **BiInt**. Therefore, failure of these properties would represent a problematic deviation from these logics, since our intention is for **DCInt** to remain similar to **BiInt** while it obtains the disjunction property. In this sense the properties can be considered part of the definition of **DCInt**. Both can be shown to hold by a straightforward induction proof.

The verification of an implication and the falsification of an exclusion in **DCInt** semantics is similar to the interpretations found in **BiInt**. In contrast, the most distinctive aspect of **DCInt** semantics {15} is its interpretation of the verification of an exclusion formula, and the falsification of an implication formula. The left half of Figure 9 diagrams the interpretation of the verification of $A \to_- B$ at a world w of M, which depends on the existence of a backwardly reachable world $w' \in \uparrow_- w$; dually, the right half diagrams the $A \to_+ B$ case, which is perfectly symmetric to the $A \to_- B$ case. The world w' in turn

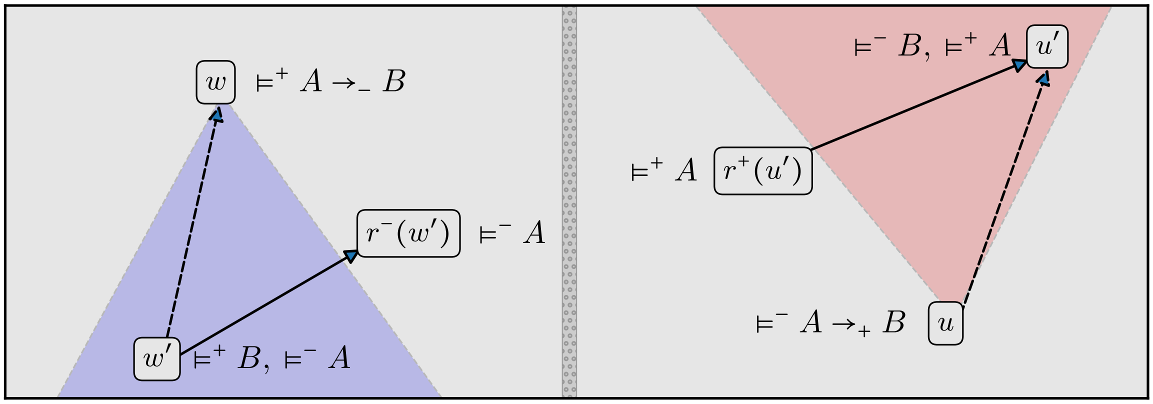


Figure 9: The interpretation of $A \to_p B$ at a world of **DCInt** semantics, where the right side depicts the case of $p = +$ at a world u and the left depicts the case of $p = -$ at a world w (the following description applies to the left diagram only, but the right diagram can be described dually). The dashed line for the arrow between w and w' represents the fact that w' exists due to the assumption that $w \vDash^+ A \to_- B$. The wedge indicates the worlds that are negatively reachable from w. The world $r^-(w')$ is outside of the wedge because it does not necessarily need to be reachable from w.

determines a dual counterpart world $r^-(w')$ which must be in the direction of positive reachability from w'; i.e. Definition 12.3 requires that $r^-(w') \in \uparrow_+ w'$. For the exclusion to be verified at w we must have that the dual counterpart world $r^-(w')$ falsifies A (i.e. $M, r^-(w') \vDash^- A$). By polar persistence {18}, if we have $M, r^-(w') \vDash^- A$ then we also have $M, w' \vDash^- A$. This means that the dual counterpart world $r^-(w')$ falsifies a subset of the formulas that w' falsifies (i.e. the set $\mathbb{T}(M, r^-(w'), -)$ is a subset of $\mathbb{T}(M, w', -)$). Additionally, $M, w' \vDash^- A$ with polar consistence {18} implies that we have $M, w' \nvDash^+ A$, and so we have a statement that corresponds to that of $B \prec A$ verified in **BiInt**: there exists $w' \leq w$ such that $M, w' \vDash^+ B$ and $M, w' \nvDash^+ A$. In this sense the dual counterpart world is negatively weaker than w', and so the **DCInt** interpretation of a verified exclusion can be seen as a strengthening of the **BiInt** interpretation of a verified exclusion from Figure 5d. This is a stronger condition because it is possible to falsify A at w', but not falsify it at the dual counterpart world. This highlights one important role that the dual counterpart function plays in **DCInt** semantics: it makes exclusions and implications harder to verify and falsify, respectively. This effect of the dual counterpart function is instrumental in the proof that **DCInt** has the disjunction property. Finally, note that the definition of a verification of a dual-intuitionistic negation in **DCInt** follows from the definition of a verification of an exclusion: a world w verifies $\urcorner A$ iff there exists $w' \in \uparrow_- w$ such that $r^-(w')$ falsifies A.

Note that since $r^-(w')$ must be a member of $\uparrow_+ w'$ it is illustrative to observe that the use of this designated counterpart world in the interpretation of a verified exclusion is distinct from an interpretation that generally quantifies over $\uparrow_+ w'$ (and does not designate a specific counterpart world). For example, consider the result of changing the interpretation of $M, w \vDash^+ A \to_- B$ to the following: there exists $w' \in \uparrow_- w$ such that there exists $w'' \in \uparrow_+ w'$ such that $M, w'' \vDash^- A$ and $M, w' \vDash^+ B$. Let us call this statement P. Now we will show that P is logically equivalent to another statement that does not quantify over $\uparrow_+ w'$, which we will call Q: there exists $w' \in \uparrow_- w$ such that $M, w' \vDash^- A$ and $M, w' \vDash^+ B$. First note that Q implies P because w' can serve as the choice for w'' (since we have $w' \in \uparrow_+ w'$). To see that P implies Q, suppose

define term from column 1 by substituting it for "T"		
term T	ϕ is T over $\mathcal{C}$ iff ...	expression
(p)-valid	$M \vDash^p \phi$ for every structure M in $\mathcal{C}$	... iff $\mathcal{C} \vDash^p \phi$
(p)-satisfiable	$M, w \vDash^p \phi$ for some structure M in $\mathcal{C}$ and world $w \in M$	
(p)-countermodeled	ϕ is not (p)-valid over $\mathcal{C}$	... iff $\mathcal{C} \nvDash^p \phi$
(p)-unsatisfiable	ϕ is not (p)-satisfiable over $\mathcal{C}$	
(p)-contingent[8]	ϕ is (p)-satisfiable, (p)-countermodeled, and $(\overline{p})$-unsatisfiable over $\mathcal{C}$	

(10a)

define term from column 1 by substituting it for "T"	
term T	ϕ is T over $\mathcal{C}$ iff ...
contingent	for each p, ϕ is (p)-satisfiable and (p)-countermodeled over $\mathcal{C}$
valid	ϕ is $(+)$-valid over $\mathcal{C}$
verify valid	ϕ is $(+)$-valid over $\mathcal{C}$
falsify valid	ϕ is $(-)$-valid over $\mathcal{C}$
unsatisfiable	ϕ is $(+)$-unsatisfiable over $\mathcal{C}$
satisfiable	ϕ is $(+)$-satisfiable over $\mathcal{C}$
countermodeled	ϕ is $(+)$-countermodeled over $\mathcal{C}$

(10b)

Figure 10: Each table defines semantic classifications for a formula ϕ with respect to a class of **DCInt** frames $\mathcal{C}$. Each term in the leftmost column is defined by substituting it for "T" in the column to its right. Table (a) defines the classifications that are parameterized by a polarity p, and Table (b) defines non-polarized classifications using the polarized ones. Additionally, each non-empty cell of the rightmost column of Table (a) defines a symbolic expression that is equivalent to the statement in the center column.

that P is true. In this case the polar persistence property and Lemma 11 applied to the facts of $M, w'' \vDash^- A$ and $w'' \in \uparrow_+ w'$ implies that we have $M, w' \vDash^- A$. Therefore Q is true. The existential quantification in P turned out to be irrelevant because P is equivalent to Q. The next section {4.1} defines the logic–called **Partial BiInt**–that is the result of interpreting exclusion according to Q. There it is shown that **Partial BiInt** lacks the disjunction property.

We conclude this section by formally defining **DCInt** {20} and discussing the polarized notions of valid, satisfiable, countermodeled, and unsatisfiable {19}.

Definition 19. *Figure 10 defines a collection of semantic classifications for **DCInt** formulas. Each term assumes the class of all **DCInt** frames when no class is mentioned. Additionally, for each **BiInt** formula ϕ, define the following for each term from the figure substituted for* T*: ϕ is* T *in **DCInt** over $\mathcal{C}$ iff $\tau(\phi)$ is* T *over $\mathcal{C}$.*

DCInt is formally defined as the set of formulas (over Σ variables) that are $(+)$-valid. This definition is sufficient for our current setting because we are not investigating the proof theory for **DCInt**.

[8] We borrow the term "contingent" from modal logic, where a formula is called contingent iff it is neither necessary nor impossible (see page 15 of [21]).

		sat.	valid		sat.		
#	possible	+	+	−	−	comment	example
0	no	✗	✗	✗	✗	Theorem 40	
1	yes	✗	✗	✗	✓	(−)-contingent	$\sigma \wedge_+ \neg_+ \sigma$
2	no	✗	✗	✓	✗	(−)-valid implies (−)-satisfiable	
3	yes	✗	✗	✓	✓	(−)-valid	$\sigma \rightarrow_- \sigma$
8	yes	✓	✗	✗	✗	(+)-contingent	$\sigma \wedge_- \neg_- \sigma$
9	yes	✓	✗	✗	✓	contingent	σ
11	no	✓	✗	✓	✓	Theorem 21 implies (+)-unsatisfiable	
12	yes	✓	✓	✗	✗	(+)-valid	$\sigma \rightarrow_+ \sigma$
13	no	✓	✓	✗	✓	Theorem 21 implies (−)-unsatisfiable	
15	no	✓	✓	✓	✓	Theorem 21	

Figure 11: The possibilities for the semantic classification of a **DCInt** formula. The # column contains the binary number associated with four bits representing the four boolean statements "the formula is (+)-satisfiable ((+)-valid, (−)-valid, and (−)-satisfiable, respectively)". Rows #4, #5, #6, #7, #10, and #14 are omitted because they have the value "no" in Column 2 for an analogous reason as Row #2.

Definition 20. *For each Σ, define* ***DCInt*** *over Σ as the set $\{\phi \in \mathbb{F}(\Sigma) \mid \phi$ is (+)-valid$\}$.*

The independence of the verification and falsification status of a formula at a world (see Remark 17) extends to the polarized notions of validity and satisfiability, and so a formula can be (−)-unsatisfiable but also not (+)-valid. Figure 11 enumerates the feasible combinations of (+)-satisfiable, (+)-valid, (−)-valid, and (−)-satisfiable. It shows that the **DCInt** notions of verification and falsification are logically opposed to each other in the sense that every formula of the language can be categorized as exactly one of (+)-contingent, (+)-valid, (−)-valid, (−)-contingent, or contingent. The contrapositive of Theorem 21 shows that these notions of validity and satisfiability respect each other in the sense that if a formula is ($\overline{p}$)-satisfiable then it is (p)-countermodeled. This comports with the intuitive understanding of the meaning of verification and falsification since–for example–this means that exhibiting a world that falsifies a formula ϕ (i.e. it is (−)-satisfiable) is sufficient to demonstrate that ϕ is not verify valid (i.e. it is (+)-countermodeled).

Theorem 21. *For every formula ϕ and polarity p: if ϕ is (p)-valid, then ϕ is ($\overline{p}$)-unsatisfiable.*

Proof. The conclusion follows directly from the polar consistence property {18}. □

In accordance with Remark 17, the converse of Theorem 21 is not true: $\sigma \wedge_+ \neg_+ \sigma$ is (+)-unsatisfiable but is not (−)-valid. This formula is shown in row #1 of Figure 11 as an example of a (−)-contingent formula. In section 4.4 we prove Theorem 40, which is a weaker form of the converse of Theorem 21. It establishes that every formula of **DCInt** must be either (+)-satisfiable or (−)-satisfiable, which justifies row #0 of Figure 11.

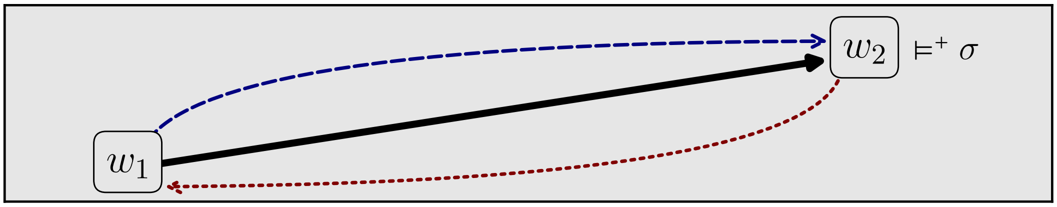
4.1 Partial BiInt

The precise relationship between both **Int** and **BiInt** Kripke semantics, and **DCInt** Kripke semantics is not discussed until Section 4.4. However, it is accurate to say that the **DCInt** interpretation of a verified implication, verified conjunction, falsified exclusion, and a falsified disjunction is roughly equiva-

$$M, w \vDash^+_{\mathbf{D}} B \prec A \text{ iff there exists } w' \leq w \text{ such that } M, w' \vDash^+_{\mathbf{D}} B \text{ and } M, w' \vDash^-_{\mathbf{D}} A \tag{5}$$

$$M, w \vDash^+_{\mathbf{D}} {\sim} A \text{ iff there exists } w' \leq w \text{ such that } M, w' \vDash^-_{\mathbf{D}} A \tag{6}$$

(12a) Interpretation of a verified exclusion and dual-intuitionistic negation in **Partial BiInt**



(12b) **DCInt** countermodel of $\neg\sigma \vee {\sim}\neg\sigma$. The blue dashed line indicates the negative dual counterpart, and the red dotted line indicates the positive dual counterpart.

lent to the **BiInt** interpretation. Thus the schema $A \to A$ is (+)-valid in **DCInt**, and the schema $A \wedge \neg A$ is (+)-unsatisfiable; and–dually–the schema $A \prec A$ is (−)-valid in **DCInt**, and the schema $A \vee {\sim} A$ is (−)-unsatisfiable (in the polarized language these four schemas are $A \to_+ A$, $A \wedge_+ \neg_+ A$, $A \to_- A$, and $A \wedge_- \neg_- A$). As noted in the previous section, **DCInt** semantics is different from **BiInt** in two important ways: **(1)** it is partial (see Remark 17); and **(2)** it interprets falsified implication and verified exclusion differently. The purpose of this section is to explain why these are both important.

As a result of the first difference, a formula such as $\sigma \vee {\sim}\sigma$ is valid in **BiInt** but not in **DCInt**.

Theorem 22. *For any proposition σ, $\sigma \vee {\sim}\sigma$ is (+)-countermodeled in **DCInt**.*

Proof. This formula fails to be verified by a world w in a structure where that is the only world, and the valuations v^+ and v^- are such that $v^+(\sigma) = v^-(\sigma) = \varnothing$. The formula ${\sim}\sigma$ is not verified at w because there is only one negatively reachable world, namely w itself; and in this negatively reachable world the negative counterpart is again w because that is the only choice; finally, this negative counterpart does not falsify σ. □

In the discussion around Theorem 8, this was the formula that we used to demonstrate that **BiInt** lacks the disjunction property. Since the countermodel of this formula did not involve the "dual counterpart" aspect of **DCInt** semantics, it is natural to question whether that aspect of the semantics is actually necessary for the disjunction property. We will now briefly consider an alternative semantics in order to show why the aspect of partiality in **DCInt** semantics is not by itself sufficient to obtain the disjunction property. By Lemma 13, a structure can always choose w' itself as the dual counterpart $r^-(w')$ (i.e. define $r^-(w') = w'$). In this case the interpretations of exclusion and dual-intuitionistic negation change to the statements shown in Figure 12a. The class of **DCInt** frames that interpret exclusion as shown in Equivalence 5 determines a Kripke semantics {23} that is like **BiInt** semantics, but where formulas are partially interpreted. However, Theorem 24 shows that modifying **BiInt** semantics with partial interpretation in this way is not enough to obtain the disjunction property.

Definition 23 (**Partial BiInt**)**.** *A **DCInt** frame $\langle W, \leq, r^+, r^- \rangle$ is in the class $\mathcal{C}_{P.B.}$ iff both r^+ and r^- are equal to the identity function on W. Define the logic **Partial BiInt** by the semantics of **DCInt**, but where every frame must be a member of $\mathcal{C}_{P.B.}$.*

Theorem 24. ***Partial BiInt** does not have the disjunction property.*

Proof. The formula $\neg\sigma \vee \sim\neg\sigma$ is (+)-valid in **DCInt** over $\mathcal{C}_{\text{P.B.}}$; and neither $\neg\sigma$ nor $\sim\neg\sigma$ is (+)-valid in **DCInt** over $\mathcal{C}_{\text{P.B.}}$. To see why the statement for $\neg\sigma \vee \sim\neg\sigma$ holds, consider a structure with a frame from $\mathcal{C}_{\text{P.B.}}$, let w be a world from that structure, and suppose that $w \nvDash^{+}_{\mathbf{D}} \neg\sigma$. This implies that there exists a world $w' \geq w$ such that we have $w' \vDash^{+}_{\mathbf{D}} \sigma$. Now we can show that we have $w \vDash^{+}_{\mathbf{D}} \sim\neg\sigma$ by showing that there exists a world $w'' \leq w$ such that $w'' \vDash^{-}_{\mathbf{D}} \neg\sigma$. Choosing w itself for w'', we must check that there exists a world $w''' \geq w$ such that $w''' \vDash^{+}_{\mathbf{D}} \sigma$. By our assumption above, this clearly holds by choosing w' for w'''. □

Later on we will prove that **DCInt** has the disjunction property, and the proof critically relies on the way that **DCInt** interprets verified exclusion and falsified implication. Theorem 25 hints at the intuition for why that difference is critical because it shows how the dual counterpart functions facilitate a countermodel to the formula $\neg\sigma \vee \sim\neg\sigma$. In summary, of the two primary differences from **BiInt**: the first is not sufficient to obtain the disjunction property, but the second is; and furthermore, both are needed to obtain duality and the disjunction property together.

Theorem 25. *For any σ, the formula $\neg\sigma \vee \sim\neg\sigma$ is (+)-countermodeled in **DCInt**.*

Proof. A diagram of the countermodel is shown in Figure 12b. The structure has only the worlds w_1 and w_2, such that $w_1 \leq w_2$, $w_2 \nleq w_1$, $v^{+}(\sigma) = \{w_2\}$, $r^{-}(w_1) = w_2$, and $r^{+}(w_2) = w_1$. Clearly we have $w_1 \nvDash^{+}_{\mathbf{D}} \neg\sigma$, so we only need to check that we also have $w_1 \nvDash^{+}_{\mathbf{D}} \sim\neg\sigma$. By construction we have $\uparrow_{-} w_1 = \{w_1\}$, so it will suffice to show that we have $r^{-}(w_1) \nvDash^{-}_{\mathbf{D}} \neg\sigma$. Also by construction we have $\uparrow_{+} w_2 = \{w_2\}$, so it will suffice to show that we have $r^{+}(w_2) \nvDash^{+}_{\mathbf{D}} \sigma$. This clearly holds because $r^{+}(w_2) = w_1$. □

4.2 Exclusion in DCInt

In order to improve the reader's intuition for exclusion in **DCInt** this section discusses the (+)-valid status of some schemas involving exclusion (they are displayed using the **BiInt** language). Theorem 26 demonstrates that **DCInt** rejects all of Rauszer's **BiInt** exclusion axioms (shown in Figure 13a). These are the axioms A11-A14 from Rauszer's Hilbert-style axiomatization of **BiInt** (page 18 of [27]). In that system they comprise all of the axioms that characterize the exclusion connective, since axioms A1-A10 are **Int** valid and axioms A15-A18 concern the dual negation connective. In Section 3 of [29], Sano & Stell provide an alternative Hilbert-style axiomatization that has only two axioms involving exclusion: the first is equivalent to Rauszer's A11; and the second is the axiom $((A \vee B) \prec A) \rightarrow B$, which is valid in **DCInt**.

Theorem 26. *Each schema in Figure 13a is countermodeled in **DCInt** for some choice of A, B, C.*

Proof. Let σ_0 and σ_1 be propositional variables, and for each corresponding part choose **(A11)** $B = \top$, $A = \sigma_0$; and **(A12)** $B = \sigma_0, A = \sigma_1$; and **(A13)** $C = \top, B = \sigma_0, A = \sigma_1$; and **(A14)** $B = \top, A = \sigma_0$. Parts (A11) and (A14) can be countermodeled with a simple structure of one world. Part (A12) can be countermodeled with the structure shown on the left side of Figure 13b, where the positive counterpart of w_1 is w_2 and the negative counterpart of w_0 is w_1. Part (A13) can be countermodeled with the structure shown on the right side of Figure 13b, where for the negative counterpart function r^{-} we have $r^{-}(w_0) = w_1$ and $r^{-}(w_2) = w_3$. □

A11. $B \to (A \vee (B \prec A))$

A12. $(B \prec A) \to {\sim}(B \to A)$

A13. $((C \prec B) \prec A) \to (C \prec (B \vee A))$

A14. $\neg(B \prec A) \to (B \to A)$

(13a)

(13b)

Figure 13: Subfigure (a) contains schemas that are (+)-countermodeled in **DCInt**, and Subfigure (b) displays structures for the proof of Theorem 26.

Though **DCInt** rejects axioms A11-A14, it does maintain compatibility with Rauszer's "r" rule from page 19 of [27]. Additionally, it maintains compatibility with a rule named "Mon$\prec$" from Sano & Stell's axiomatization of **BiInt**. Theorem 27 shows that these rules are sound with respect to **DCInt** semantics by rephrasing them in terms of validity.[9]

Theorem 27. *For **BiInt** formulas A,B,C:*

1. *("r") if A is valid in **DCInt** then $\neg{\sim}A$ is valid in **DCInt***
2. *("Mon$\prec$") if $A \to B$ is valid in **DCInt** then $(A \prec C) \to (B \prec C)$ is valid in **DCInt***
3. *if $C \to (A \vee B)$ is valid in **DCInt** then $(C \prec A) \to B$ is valid in **DCInt***

Proof. For part (1), suppose A is valid. Then A is (−)-unsatisfiable by Theorem 21. Therefore $\neg{\sim}A$ is valid, since no world w of any structure can verify ${\sim}A$: if $w \vDash^+_{\mathbf{D}} {\sim}A$ then there exists $w' \leq w$ such that $r^-(w') \vDash^-_{\mathbf{D}} A$. For part (2), suppose $A \to B$ is valid and that a world w of an arbitrary structure verifies $A \prec C$. This means that there exists $w' \leq w$ such that $w' \vDash^+_{\mathbf{D}} A$ and $r^-(w') \vDash^-_{\mathbf{D}} C$. We have $w' \vDash^+_{\mathbf{D}} B$ because $A \to B$ is valid, and so this implies $w \vDash^+_{\mathbf{D}} B \prec C$. w was arbitrary, so $(A \prec C) \to (B \prec C)$ is valid. Part (3) follows by a semantic analysis that is similar to the first two parts. □

The third part of the theorem is related to a property about **BiInt** exclusion that Wansing [35] points out, which is that exclusion is the residuum of additive disjunction. This is expressed in the following statement involving the **BiInt** entailment relation: $C \vdash A \vee B$ iff $C \prec A \vdash B$. It is notable that this fact will not hold for an entailment relation of **DCInt** because the right-to-left direction of the equivalence will fail. Choosing $C = \top$, $A = \sigma$, and $B = {\sim}\sigma$, we have the statements $\top \prec \sigma \vdash {\sim}\sigma$ and $\top \vdash \sigma \vee {\sim}\sigma$. For an entailment relation of **DCInt** the former inference will be true because of the definition of ${\sim}$, while Theorem 22 implies that the latter cannot be true. However, part (3) of Theorem 27 shows that the left-to-right direction of the equivalence will hold.

[9] In [18], Goré & Shillito resolve a significant amount of confusion that has arisen from the interpretation of rule "r" with respect to the Hilbert-style proof system for **BiInt**. They also explicitly point out that the same interpretation issue arises with Sano & Stell's rule. Theorem 27 interprets the rules in the way that Goré & Shillito describe as "weak". We thank one of the reviewers for drawing our attention to the "r" rule and the work in [29] and [18].

1. $\neg(A \prec A)$
2. $(C \prec ((B \prec A) \vee A)) \to (C \prec (B \vee A))$
3. $(C \prec (B \vee A)) \to ((C \prec B) \prec A)$
4. $((C \prec B) \prec A) \to ((C \prec B) \wedge (C \prec A))$
5. $(B \prec A) \to (B \wedge {\sim}A)$
6. $((B \vee A) \prec A) \to (B \prec A)$
7. $(B \prec (B \wedge A)) \to (B \prec A)$
8. ${\sim}(B \to A) \to (B \prec A)$

Figure 14: Valid schemas of **DCInt**

Finally, this section concludes with some examples of schemas involving exclusion that are valid in **DCInt** {28}. Note that schema 3 is the converse of A13, schema 8 is the converse of A12, and that schema 6 is essentially a stronger version of Sano & Stell's second axiom. Though we are leaving the axiomatic characterisation of **DCInt** exclusion to future work, we believe that these schemas are useful for gaining an intuition for it.

Theorem 28. *For **DCInt** formulas A, B, C, the formulas of Figure 14 are valid in **DCInt**. Also, the schema $((A \vee B) \prec A) \to B$ is valid (Sano & Stell's second exclusion axiom from [29]).*

Proof. For part (1), suppose for sake of contradiction that there exists a structure M with negative counterpart function r^- and world $w_0 \in M$ such that $M, w_0 \nvDash^+_{\mathbf{D}} \neg(A \prec A)$. By definition this means that there exists $w_1 \in {\uparrow_+} w_0$ such that $M, w_1 \vDash^+_{\mathbf{D}} A \prec A$, which in turn means that there exists $w_2 \in {\uparrow_-} w_1$ such that $M, r^-(w_2) \vDash^-_{\mathbf{D}} A$ and $M, w_2 \vDash^+_{\mathbf{D}} A$. By definition we have $r^-(w_2) \in {\uparrow_+} w_2$, so the polar persistence property {18} implies that we have $M, r^-(w_2) \vDash^+_{\mathbf{D}} A$. This contradicts the polar consistence property {18}. Parts (2) through (8) follow by similar analysis of **DCInt** semantics. □

The first schema of Figure 14 is valid because of the polar consistence property {18}, and that one essentially shows that the verifier and falsifier must be consistent with each other. The intuition for the other schemas can be similarly summarized in terms of "interaction" between the verifier and falsifier. These examples were chosen because they demonstrate situations in which the verifier discovers information from the falsifier. We believe that the reader will readily gain such intuitions by proving the validity of each schema via an analysis of the semantics (especially if the reader constructs a world diagram similar to that of Figure 9, 12b, or 13b). For example, schema 2 essentially shows that the verifier can observe the consequence of a falsified exclusion: if there is a world w' that verifies C and the world $r^-(w')$ falsifies both $B \prec A$ and A, then we know that w' verifies C and $r^-(w')$ falsifies both B and A.

4.3 The duality of DCInt

In this section we specify the duality correspondence for **DCInt** formulas {29}, and prove that **DCInt** has a duality property that is analogous to the duality of **CL**.

Definition 29 (dual formulas, frames, and structures)**.** *The dualization function $\delta_{\mathbf{D}}$ is defined on **DCInt** formulas by Figure 15a, and **DCInt** frames and structures by Figure 15b.*

Lemma 30 establishes some basic properties of $\delta_{\mathbf{D}}$ and states that the duality correspondence of **DCInt** is equivalent to that of **BiInt**.

Lemma 30. *The function $\delta_{\mathbf{D}}$ is an involution that maps formulas to formulas, frames to frames, and structures to structures. Also, for $\delta_{\mathbf{D}}$ on formulas, we have $\delta_{\mathbf{D}} = \tau \circ \delta_{\mathbf{B}} \circ \tau^{-1}$.*

$$\delta_{\mathbf{D}}(\sigma) = \sigma \qquad \delta_{\mathbf{D}}(A \wedge_p B) = \delta_{\mathbf{D}}(A) \wedge_{\overline{p}} \delta_{\mathbf{D}}(B)$$
$$\delta_{\mathbf{D}}(\top_p) = \top_{\overline{p}} \qquad \delta_{\mathbf{D}}(A \to_p B) = \delta_{\mathbf{D}}(A) \to_{\overline{p}} \delta_{\mathbf{D}}(B)$$

(15a)

$$\delta_{\mathbf{D}}(\langle W, \leq, r^+, r^- \rangle) = \langle W, \leq^{-1}, r^-, r^+ \rangle$$
$$\delta_{\mathbf{D}}(\langle F, v^+, v^- \rangle) = \langle \delta_{\mathbf{D}}(F), v^-, v^+ \rangle$$

(15b)

Figure 15: The duality correspondence of **DCInt** is given in subfigure (a), and subfigure (b) defines $\delta_{\mathbf{D}}$ on **DCInt** frames and structures.

The world duality theorem {31} is analogous to Theorem 2 of **CL** semantics. This theorem and Lemma 30 each follow from just checking definitions, using the polar reachability lemma {11}, and by induction on the complexity of the formula ϕ.

Theorem 31 (world duality)**.** *For every polarity p, **DCInt** structure M, formula ϕ, and $w \in M$: we have $M, w \vDash^p \phi$ iff $\delta_{\mathbf{D}}(M), w \vDash^{\overline{p}} \delta_{\mathbf{D}}(\phi)$.*

Finally, the duality theorem {32} states that **DCInt** has duality with respect to validity and satisfiability, which is analogous to Theorem 3. In the next section we will prove that **DCInt** is a conservative extension of **Int** and later on we will prove that it has the disjunction property. Recall that Remark 17 pointed out that the semantics of **DCInt** distinguishes between the notions of (+)-unsatisfiable and (−)-valid. This means that the argument used in Theorem 8 does not apply to **DCInt**, even though it has both duality and the disjunction property.

Theorem 32 (duality of **DCInt**)**.** *For every polarity p and formula ϕ:*

1. *ϕ is (p)-valid iff $\delta_{\mathbf{D}}(\phi)$ is $(\overline{p})$-valid*
2. *ϕ is (p)-satisfiable iff $\delta_{\mathbf{D}}(\phi)$ is $(\overline{p})$-satisfiable*
3. *if ϕ is (p)-valid then $\delta_{\mathbf{D}}(\phi)$ is (p)-unsatisfiable*

Proof. Parts 32.1 and 32.2 follow directly from the previous lemmas. To prove Part 32.3, suppose that ϕ is (p)-valid. Part 32.1 implies that $\delta_{\mathbf{D}}(\phi)$ is $(\overline{p})$-valid, and so therefore Theorem 21 implies that $\delta_{\mathbf{D}}(\phi)$ is (p)-unsatisfiable. □

4.4 Relating DCInt to Int, DualInt, and BiInt

DCInt is a conservative extension of **Int** and is a sublogic of **BiInt** because **Int** structures {5} can be faithfully represented as **DCInt** Kripke structures, and vice versa. We recall the definitions of conservative extensions and sublogics in Definition 33.

Definition 33. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be the sets of formulas of the languages in which logics $\mathbf{L}_1$ and $\mathbf{L}_2$ are respectively expressed, where both languages are generated by the same propositional variables. Also let $\mathcal{L}_1 \subseteq \mathcal{L}_2$, and T_1 and T_2 be the set of theorems of $\mathbf{L}_1$ and $\mathbf{L}_2$, respectively.* ***(1)*** *$\mathbf{L}_2$ is an extension of $\mathbf{L}_1$ iff $\mathbf{L}_1$ is a sublogic of $\mathbf{L}_2$ iff $T_1 \subseteq T_2$; and* ***(2)*** *$\mathbf{L}_2$ is a conservative extension of $\mathbf{L}_1$ iff $T_2 \cap \mathcal{L}_1 = T_1$.*

The functions $\mu_{\mathbf{I}}$, $\mu_{\mathbf{U}}$, $\mu^+_{\mathbf{D}}$ and $\mu^-_{\mathbf{D}}$ {34} specify the mappings between the two kinds of structures. The first translation lemma {35} establishes some important basic facts about these functions, including their relationship to each other and to the functions $\delta_{\mathbf{D}}$ and $\delta_{\mathbf{B}}$.

Definition 34. *Let $M = \langle W, \leq, r^+, r^-, v^+, v^- \rangle$ be a **DCInt** structure. Define the functions μ_I and μ_U by $\mu_I(M) = \langle W, \leq, v^+ \rangle$, and $\mu_U(M) = \langle W, \leq^{-1}, v^- \rangle$.*

*Let $M = \langle W, \leq, v \rangle$ be an **Int** structure {5}. Define the functions μ_D^+ and μ_D^- by $\mu_D^+(M) = \langle W, \leq, \iota_W, \iota_W, v, \overline{v} \rangle$, and $\mu_D^-(M) = \langle W, \leq^{-1}, \iota_W, \iota_W, \overline{v}, v \rangle$, where ι_W is the identity function on W, and $\overline{v} = \sigma \mapsto W \setminus v(\sigma)$.*

Lemma 35 (translation I)**.** *1. Every tuple in the image of μ_I and μ_U is an **Int** structure {5}*

2. *every tuple in the image of μ_D^+ and μ_D^- is a **DCInt** structure {14}*
3. *both of the functions $\mu_I \circ \mu_D^+$ and $\mu_U \circ \mu_D^-$ are equal to the identity function*
4. *for each **DCInt** structure M and each polarity p: **(a)** $\delta_D(\mu_D^p(M)) = \mu_D^{\overline{p}}(M)$; and **(b)** for each **BiInt** formula ϕ: $\mu_D^+(M), w \vDash_D^p \phi$ iff $\mu_D^-(M), w \vDash_D^{\overline{p}} \delta_B(\phi)$*

Proof. Parts (1) and (2) follow from Lemmas 13 and 11, and parts (3) and (4a) follow from just checking definitions. Part (4b) can be proved using $\delta_\mathbf{D} \circ \mu_\mathbf{D}^p = \mu_\mathbf{D}^{\overline{p}}$ (4a), Theorem 31, and $\delta_\mathbf{D} \circ \tau = \tau \circ \delta_\mathbf{B}$ {30}. □

Theorem 37 proves that **DCInt** conservatively extends **Int** with respect to verify validity, and also that it conservatively extends **DualInt** with respect to falsify validity (i.e. unsatisfiability). Note that it does not extend **Int** with respect to falsify validity, nor **DualInt** with respect to verify validity (for example, $\sigma \wedge \neg\sigma$ is falsify valid in **Int**, and $\sigma \vee {\sim}\sigma$ is verify valid in **DualInt**). Theorem 37 relies on the second translation lemma {36}. That lemma follows from a straightforward induction proof on ϕ, and states that the functions $\mu_\mathbf{I}$ and $\mu_\mathbf{U}$ are faithful with respect to ϕ.

Lemma 36 (translation II)**.** *For every **DCInt** structure M and world $w \in M$:*

1. *if ϕ is an **Int** formula, then $M, w \vDash_D^+ \phi$ iff $\mu_I(M), w \vDash_I \phi$*
2. *if ϕ is a **DualInt** formula, then $M, w \vDash_D^- \phi$ iff $\mu_U(M), w \nvDash_U \phi$*

Theorem 37 (**DCInt** conservatively extends **Int**)**.**

1. (a) *An **Int** formula is valid in **Int** iff it is (+)-valid in **DCInt***
   (b) *An **Int** formula is satisfiable in **Int** iff it is (+)-satisfiable in **DCInt***
2. (a) *A **DualInt** formula is unsatisfiable in **DualInt** iff it is (−)-valid in **DCInt***
   (b) *A **DualInt** formula is countermodeled in **DualInt** iff it is (−)-satisfiable in **DCInt***

Proof. For part (1), let ϕ be an **Int** formula. There are four implications, and we sketch their proof in two separate pairs (in each case the first of the pair is proved by contrapositive). Part (2) follows similarly.

The first pair: if $\tau(\phi)$ is (+)-valid then ϕ is **Int** valid; and if ϕ is **Int** satisfiable then $\tau(\phi)$ is (+)-satisfiable. These two implications follow from the first two translation lemmas. For any **Int** structure M, translation I {35} implies that we have $\mu_\mathbf{I}(\mu_\mathbf{D}^+(M)) = M$; and translation II {36} implies that for every $w \in M$ we have: $M, w \vDash_\mathbf{I} \phi$ iff $\mu_\mathbf{I}(\mu_\mathbf{D}^+(M)), w \vDash_\mathbf{I} \phi$ iff $\mu_\mathbf{D}^+(M), w \vDash^+ \tau(\phi)$. Therefore an **Int** countermodel induces a positive **DCInt** countermodel, and an **Int** model induces a positive **DCInt** model.

The second pair: if ϕ is **Int** valid then $\tau(\phi)$ is (+)-valid; and if $\tau(\phi)$ is (+)-satisfiable then ϕ is **Int** satisfiable. These two implications follow by an argument similar to that of the first pair. □

The third translation lemma {38} states that $\mu_\mathbf{D}^+$ and $\mu_\mathbf{D}^-$ are faithful with respect to **BiInt** formulas, which implies that **DCInt** is a sublogic of **BiInt** {39}.

Lemma 38 (translation III)**.** *For every **Int** structure M, world $w \in M$, and **BiInt** formula ϕ:*

1. $M,w \vDash_{\boldsymbol{B}} \phi$ *iff* $\mu^{+}_{\boldsymbol{D}}(M),w \vDash^{+}_{\boldsymbol{D}} \phi$
2. $M,w \nvDash_{\boldsymbol{B}} \phi$ *iff* $\mu^{+}_{\boldsymbol{D}}(M),w \vDash^{-}_{\boldsymbol{D}} \phi$
3. $M,w \vDash_{\boldsymbol{B}} \phi$ *iff* $\mu^{-}_{\boldsymbol{D}}(M),w \vDash^{-}_{\boldsymbol{D}} \delta_{\boldsymbol{B}}(\phi)$
4. $M,w \nvDash_{\boldsymbol{B}} \phi$ *iff* $\mu^{-}_{\boldsymbol{D}}(M),w \vDash^{+}_{\boldsymbol{D}} \delta_{\boldsymbol{B}}(\phi)$

Proof. By Part (4b) of translation I $\{35\}$, Parts (3) and (4) follow from (1) and (2). Parts (1) and (2) can be proved simultaneously by a straightforward induction proof on ϕ. □

Theorem 39 (**DCInt** is a sublogic of **BiInt**)**.** *For every **BiInt** formula ϕ we have:*

1. *(a) If ϕ is (+)-valid in **DCInt**, then ϕ is **BiInt** valid*
   *(b) If ϕ is (−)-valid in **DCInt**, then ϕ is **BiInt** unsatisfiable*
2. *(a) If ϕ is **BiInt** satisfiable then ϕ is (+)-satisfiable in **DCInt***
   *(b) If ϕ is **BiInt** countermodeled, then ϕ is (−)-satisfiable in **DCInt***

Proof. We prove the contrapositive of Part (1a). The rest of the implications can be proved through similar arguments. Suppose that M is a **BiInt** structure with $w \in M$ such that $M,w \nvDash_{\mathbf{B}} \phi$. By the third translation lemma $\{38\}$ we have $\mu^{+}_{\mathbf{D}}(M),w \nvDash^{+} \tau(\phi)$, so therefore $\tau(\phi)$ is not (+)-valid. □

Finally, Theorem 40 follows as a corollary to Theorem 39. This theorem was used to justify the first row of the table in Figure 11.

Theorem 40. *For every formula ϕ and polarity p: if ϕ is (p)-unsatisfiable then ϕ is $(\overline{p})$-satisfiable.*

Proof. First note that $\tau^{-1}(\phi)$ must either be **BiInt** satisfiable or **BiInt** countermodeled. Now we cannot have that ϕ is both (+)-unsatisfiable and (−)-unsatisfiable, since this contradicts part 2 of Theorem 39. □

4.5 DCInt bisimulations

The notion of a bisimulation between two structures of a Kripke semantics is useful because it establishes a condition under which two worlds of different structures are logically equivalent. **Int** bisimulations are developed in [22] (Section 5), [25], and [16]; and **BiInt** bisimulations are developed in [1] and [19]. Similar ideas also appear in the context of modal logic, such as in [2] and [3]. In this section we define the notion of bisimulation that corresponds to **DCInt** structures, and develop some results about bisimulations that will be useful for proving the disjunction property in Section 5.

Definition 41. *For sets X, Y, and $X' \subseteq X$, and relation $R \subseteq X \times Y$, define the image of X' under R as the set $R[X'] = \{y \in Y \mid \text{there exists } x \in X' \text{ such that } xRy\}$. For $x \in X$, define $R[x] = R[\{x\}]$. For set Z and relation $S \subseteq Y \times Z$, define the relational composition $R;S$ as the set $\{(x,z) \in X \times Z \mid \text{there exists } y \in Y \text{ such that } xRySz\}$.*

Using the notation from Definition 41, we define a directed and polarized condition for when a world is similar to another world in a different **DCInt** frame $\{42\}$. This definition is used as a template in both the "back" and "forth" conditions in the definition of a **DCInt** bisimulation $\{43\}$. The first condition of Definition 42 (Condition 1) is analogous to a condition that also appears in the definition of an **Int** bisimulation (i.e. conditions B2 and B3 of the definition in Section 3 of [16]). Naturally, the latter condition (Condition 2) is unique to **DCInt** bisimulations since it involves the dual counterpart functions. We cannot provide an intuitive explanation of Condition 2, except to say that it should all become clear to the reader that carefully works through the proof of the bisimulation invariance theorem $\{45\}$. This condition is essential to ensure that–for example–a verified exclusion is invariant across two worlds related by a bisimulation.

Definition 42 (world similarity)**.** *Let* $F_0 = \langle W_0, \leq_0, r_0^+, r_0^- \rangle$ *and* $F_1 = \langle W_1, \leq_1, r_1^+, r_1^- \rangle$ *be **DCInt** frames, and let* Z^+, $Z^- \subseteq W_0 \times W_1$, $w \in W_0$, $u \in W_1$, *and* p *be a polarity.* w is similar to u with respect to (p, F_0, F_1, Z^+, Z^-) *iff **(1)** for every* $w' \in \uparrow_p w$, *there exists* $u' \in \uparrow_p u$ *such that* $w' Z^p u'$; *and **(2)** for every* $w' \in \uparrow_{\overline{p}} w$, *there exists* $u' \in \uparrow_{\overline{p}} u$, *such that **(a)*** $(\uparrow_{\overline{p}} u') \cap (Z^p[\uparrow_p w']) \neq \varnothing$; *and **(b)*** $(\uparrow_p r_1^{\overline{p}}(u')) \cap (Z^{\overline{p}}[\uparrow_{\overline{p}} r_0^{\overline{p}}(w')]) \neq \varnothing$.

Next, we define a bisimulation as a polarized relation between two frames where for every pair of positive/negative related worlds, each world of the pair is positive/negative similar to the other $\{43\}$.

Definition 43 (frame bisimulation)**.** *Let* $F_0 = \langle W_0, \leq_0, r_0^+, r_0^- \rangle$ *and* $F_1 = \langle W_1, \leq_1, r_1^+, r_1^- \rangle$ *be **DCInt** frames, and let* Z^+, $Z^- \subseteq W_0 \times W_1$. The pair (Z^+, Z^-) is a bisimulation between F_0 and F_1 *iff for each polarity* p, *if* $w Z^p u$ *then: **(a)*** w *is similar to* u *with respect to* (p, F_0, F_1, Z^+, Z^-); *and **(b)*** u *is similar to* w *with respect to* $(p, F_1, F_0, (Z^+)^{-1}, (Z^-)^{-1})$.

We will sometimes refer to Condition 43.a as the "forth" condition, and Condition 43.b as the "back" condition. In the context of the variable names used in the bisimulation definition $\{43\}$, the "forth" condition translates verbatim to the statements shown in Definition 42. This condition is shown in a more concise form in Figure 16a, and a diagram of the second part of the condition is shown in Figure 16c. The "back" condition translates to the statement shown in Figure 16b.

Definition 44 extends Definition 43 to **DCInt** structures, so that a bisimulation between structures is any bisimulation between frames in which worlds related at polarity + verify exactly the same propositional variables, and worlds related at polarity − falsify exactly the same propositional variables. The bisimulation invariance theorem $\{45\}$ states that this agreement between worlds extends from the propositional variables to all formulas.

Definition 44 (structure bisimulation)**.** *Let* $M_0 = \langle F_0, v_0^+, v_0^- \rangle$ *and* $M_1 = \langle F_1, v_1^+, v_1^- \rangle$ *be **DCInt** structures, and let the pair* (Z^+, Z^-) *be a bisimulation between* F_0 *and* F_1. The pair (Z^+, Z^-) is a bisimulation between M_0 and M_1 *iff for each polarity* p, *if* $w Z^p u$ *then* $M_0, w \vDash^p \sigma$ *iff* $M_1, u \vDash^p \sigma$ *for each* $\sigma \in \Sigma$.

Theorem 45 (bisimulation invariance)**.** *Let* M_0 *and* M_1 *be **DCInt** structures, and let the pair* (Z^+, Z^-) *be a bisimulation between* M_0 *and* M_1. *For every* $w_0 \in M_0$, $w_1 \in M_1$, *and for each polarity* p: *if* $w_0 Z^p w_1$ *then* $\mathbb{T}(M_0, w_0, p) = \mathbb{T}(M_1, w_1, p)$.

Proof. Let $M_0 = \langle F_0, v_0^+, v_0^- \rangle$ and $M_1 = \langle F_1, v_1^+, v_1^- \rangle$ be **DCInt** structures, with $F_0 = \langle W_0, \leq_0, r_0^+, r_0^- \rangle$ and $F_1 = \langle W_1, \leq_1, r_1^+, r_1^- \rangle$. Let the pair (Z^+, Z^-) be a bisimulation between M_0 and M_1, $w_0 \in W_0$, $w_1 \in W_1$ and p be a polarity. Suppose that we have $w_0 Z^p w_1$, and let $\phi \in \mathbb{F}(\Sigma)$. The conclusion ($M_0, w_0 \vDash^p \phi$ iff $M_1, w_1 \vDash^p \phi$) holds by induction on the complexity of ϕ. The base case holds directly by the definition of a bisimulation $\{44\}$. Except for the cases of a verified exclusion or a falsified implication, the induction step is straightforward (the arguments are similar to those found in the proof of logical invariance between worlds related by an **Int** bisimulation). We only consider the case of a verified exclusion, since it is symmetric to that of a falsified implication. Furthermore, we only prove the left to right implication, since the argument for the right to left implication is symmetric. Figure 16d contains a diagram that depicts the relationships of the worlds that will be discussed in this proof.

Let $\phi \equiv A \rightarrow_- B$ and suppose that $M_0, w_0 \vDash^+ A \rightarrow_- B$, so that we want to show $M_1, w_1 \vDash^+ A \rightarrow_- B$. Since we have $M_0, w_0 \vDash^+ A \rightarrow_- B$, by definition there exists $w_0' \in \uparrow_- w_0$ such that $M_0, r_0^-(w_0') \vDash^- A$ and $M_0, w_0' \vDash^+$

w similar to u w.r.t. (p, F_0, F_1, Z^+, Z^-):

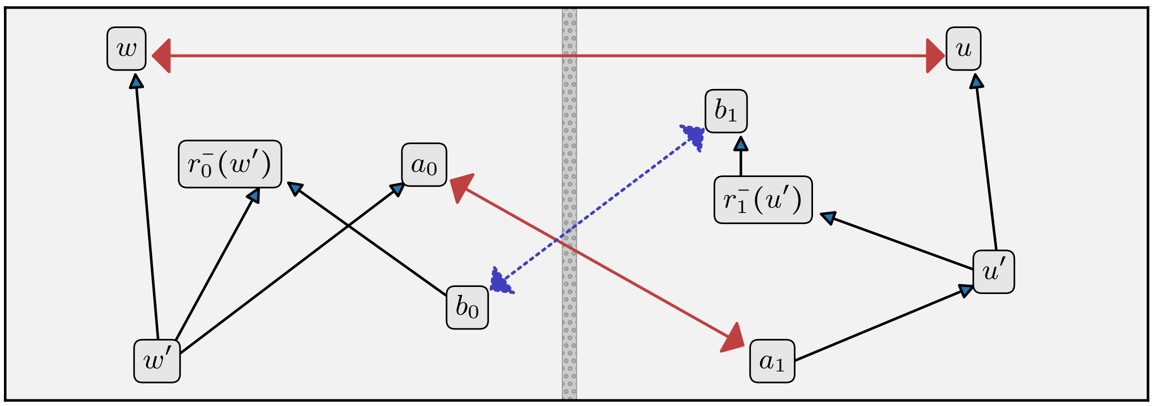
1. $\forall w' \in \uparrow_p w.\ \exists u' \in \uparrow_p u.\ w'\, Z^p\, u'$
2. $\forall w' \in \uparrow_{\overline{p}} w.\ \exists u' \in \uparrow_{\overline{p}} u.$
   (a) $\uparrow_{\overline{p}} u' \cap (Z^p[\uparrow_p w']) \neq \varnothing$
   (b) $\uparrow_p r_1^{\overline{p}}(u') \cap (Z^{\overline{p}}[\uparrow_{\overline{p}} r_0^{\overline{p}}(w')]) \neq \varnothing$

(16a) The "forth" condition

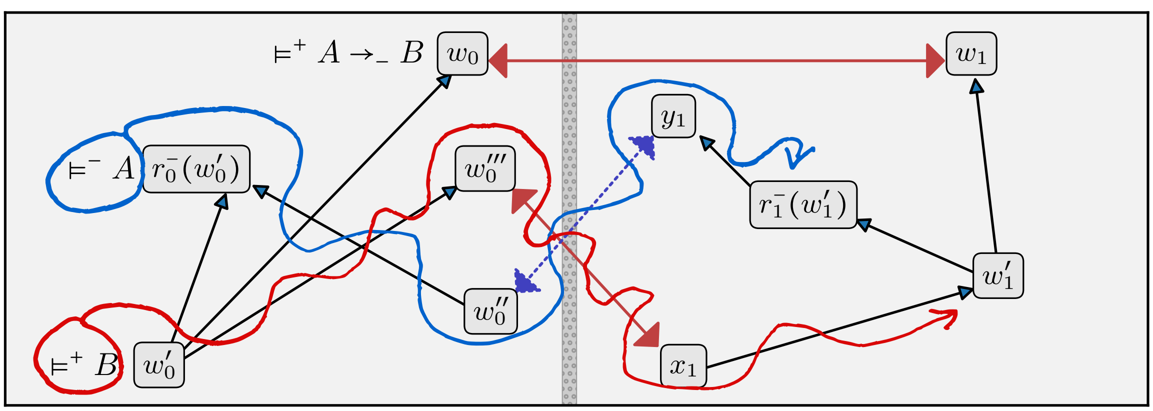
u similar to w w.r.t. $(p, F_1, F_0, (Z^+)^{-1}, (Z^-)^{-1})$:

1. $\forall u' \in \uparrow_p u.\ \exists w' \in \uparrow_p w.\ u'\, (Z^p)^{-1}\, w'$
2. $\forall u' \in \uparrow_{\overline{p}} u.\ \exists w' \in \uparrow_{\overline{p}} w.$
   (a) $\uparrow_{\overline{p}} w' \cap ((Z^p)^{-1}[\uparrow_p u']) \neq \varnothing$
   (b) $\uparrow_p r_0^{\overline{p}}(w') \cap ((Z^{\overline{p}})^{-1}[\uparrow_{\overline{p}} r_1^{\overline{p}}(u')]) \neq \varnothing$

(16b) The "back" condition



(16c) A diagram of part (2) of the "forth" condition



(16d) A diagram of a variable instantiation of part (2) of the "forth" condition

Figure 16: The (b) and (a) subfigures succinctly describe the "back" and "forth" conditions. The (c) subfigure illustrates part (2) of the "forth" condition: the a_0 and a_1 worlds must exist by part (2.a) and the b_0 and b_1 worlds must exist by part (2.b). The (d) subfigure illustrates the instantiation of part (2) of the "forth" condition that is used in the proof of Theorem 45, where the serpentine arrows indicate the path in which positive and negative formulas propagate. In subfigures (c) and (d) the thick dotted line in the center separates the worlds of the two frames, and a thin headed directed arrow between two worlds indicates that the source world reaches the target world. Also, a thick headed bidirectional arrow between two worlds indicates that the worlds are related by one of the components of the bisimulation (Z^+, Z^-), where solid red corresponds to the Z^+ component and dotted blue corresponds to the Z^- component.

B. By the "forth" property, w_0 is similar to w_1 with respect to $(+, F_0, F_1, Z^+, Z^-)$. By part 2 of this similarity property (with w_0' for w') we have: there exists $w_1' \in {\uparrow_-} w_1$ such that ${\uparrow_-} w_1' \cap (Z^+[{\uparrow_+} w_0']) \neq \varnothing$ and ${\uparrow_+} r_1^-(w_1') \cap (Z^-[{\uparrow_-} r_0^-(w_0')]) \neq \varnothing$. Therefore, let $x_1 \in {\uparrow_-} w_1' \cap (Z^+[{\uparrow_+} w_0'])$, and let $y_1 \in {\uparrow_+} r_1^-(w_1') \cap (Z^-[{\uparrow_-} r_0^-(w_0')])$. In order to show $M_1, w_1 \models^+ A \to_- B$, we will show that $M_1, r_1^-(w_1') \models^- A$ and $M_1, w_1' \models^+ B$.

Let $w_0'' \in {\uparrow_-} r_0^-(w_0')$ such that $w_0''(Z^-)y_1$, which exists because of our assumption of $y_1 \in Z^-[{\uparrow_-} r_0^-(w_0')]$. We have $M_0, w_0'' \models^- A$ by polar persistence {18} applied to $M_0, r_0^-(w_0') \models^- A$. By the induction hypothesis applied to $M_0, w_0'' \models^- A$ and $w_0''(Z^-)y_1$, we have $M_1, y_1 \models^- A$. Now note that by the polar reachability lemma {11} we have $r_1^-(w_1') \in {\uparrow_-} y_1$, since $y_1 \in {\uparrow_+} r_1^-(w_1')$. Therefore the polar persistence property implies that $M_1, r_1^-(w_1') \models^- A$.

Let $w_0''' \in {\uparrow_+} w_0'$ such that $w_0'''(Z^+)x_1$, which exists because we are assuming that $x_1 \in Z^+[{\uparrow_+} w_0']$. We have $M_0, w_0''' \models^+ B$ by polar persistence applied to $M_0, w_0' \models^+ B$. By the induction hypothesis applied to $M_0, w_0''' \models^+ B$ and $w_0'''(Z^+)x_1$, we have $M_1, x_1 \models^+ B$. The polar reachability lemma {11} implies that $w_1' \in {\uparrow_+} x_1$, since $x_1 \in {\uparrow_-} w_1'$. By polar persistence applied to this and $M_1, x_1 \models^+ B$, we have $M_1, w_1' \models^+ B$. Since $w_1' \in {\uparrow_-} w_1$, we conclude that $M_1, w_1 \models^+ A \to_- B$. ☐

Earlier we remarked that the second condition of Definition 42 lacks an intuitive explanation, but the proof argument for the invariance theorem {45} at least provides a technical explanation. In the case of a verified exclusion, this condition produces the situation shown in Figure 16d: every backwardly reachable world w_0' has a counterpart w_1' in the other structure such that $w_1' \in {\uparrow_-} w_1$; and there exists worlds y_1 and x_1 such that w_1' forwardly reaches y_1 and backwardly reaches x_1; and there exists counterpart worlds w_0'' and w_0''' such that $w_0''' Z^+ x_1$ and $w_0'' Z^- y_1$; and furthermore w_0' forwardly reaches w_0''' and $r_0^-(w_0')$ backwardly reaches w_0''. In that situation the induction hypothesis and polar persistence property {18} together imply that the verification of B at w_0' travels forward to w_0''', across to x_1, and then forward to w_1'. Similarly, the falsification of A at $r_0^-(w_0')$ travels backward to w_0'', across to y_1, and then backward to $r_1^-(w_1')$.

Next, we prove that a bisimulation is bidirectional in the sense that inverting the component relations produces another bisimulation.

Lemma 46. *Let F_0 and F_1 be **DCInt** frames. If the pair (Z^+, Z^-) is a bisimulation between F_0 and F_1, then $((Z^+)^{-1}, (Z^-)^{-1})$ is a bisimulation between F_1 and F_0.*

Proof. The conclusion follows directly from the frame bisimulation definition {43}, since conditions 43.a and 43.b are symmetric . ☐

The following two lemmas establish that the relational composition {41} of the components of two bisimulations produces another bisimulation.

Lemma 47. *Let F_0, F_1, and F_2 be **DCInt** frames, the pair (Z_0^+, Z_0^-) be a bisimulation between F_0 and F_1, and the pair (Z_1^+, Z_1^-) be a bisimulation between F_1 and F_2. Define the pair (Z^+, Z^-) by $Z^p = Z_0^p ; Z_1^p$ for each polarity p. For every $w \in F_0$, $x \in F_1$, $u \in F_2$, and for every polarity p: if we have that w is similar to x with respect to $(p, F_0, F_1, Z_0^+, Z_0^-)$, and x is similar to u with respect to $(p, F_1, F_2, Z_1^+, Z_1^-)$, then w is similar to u with respect to (p, F_0, F_2, Z^+, Z^-).*

Proof. It is easy to prove that Part (1) of the world similarity definition {42} follows directly from the definitions, so we only describe the proof for Part (2). We specifically cover the case of $p = +$, since

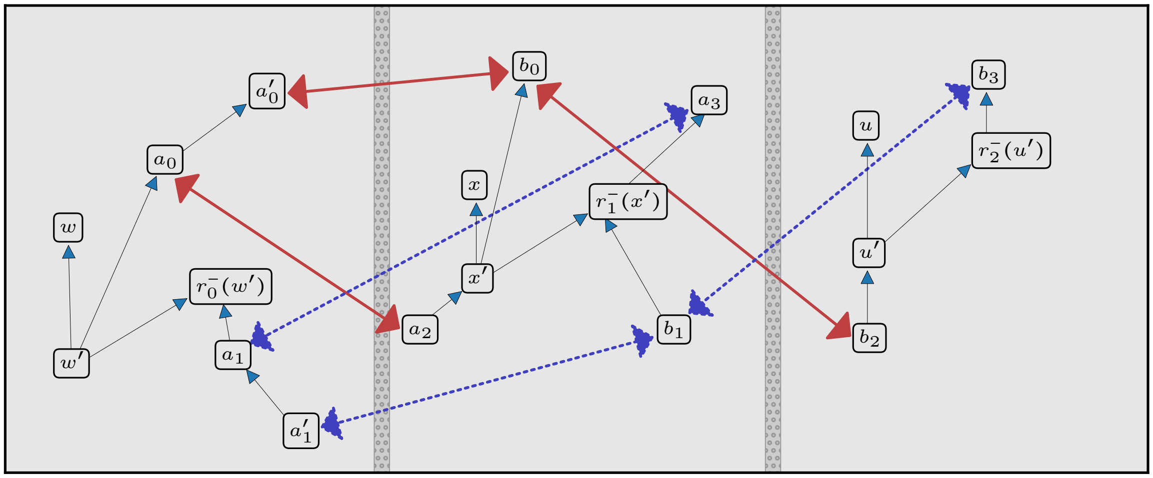


Figure 17: Diagram of the worlds that are discussed in the proof of Lemma 47, where the frames are segregated by the thick dotted lines with F_0 on the left, F_1 in the center, and F_2 on the right. The directed arrows indicate world reachability; and bidirectional arrows indicate worlds that are related by a component of the pair (Z_i^+, Z_i^-), with a solid red arrow shaft indicating Z_i^+, a dotted blue arrow shaft indicating Z_i^-, and with $i = 0$ for arrows on the left and $i = 1$ for arrows on the right.

the argument is symmetric for $p = -$. Let $w' \in \uparrow_- w$. w is similar to x with respect to $(+, F_0, F_1, Z_0^+, Z_0^-)$, so by part 2 applied to $w' \in \uparrow_- w$ we have that there exists $x' \in \uparrow_- x$ satisfying Parts 2.a and 2.b. Analogously, x is similar to u with respect to $(+, F_1, F_2, Z_1^+, Z_1^-)$, so by part 2 applied to $x' \in \uparrow_- x$ we have that there exists $u' \in \uparrow_- u$ satisfying Parts 2.a and 2.b. We want to show that **(1)** $(\uparrow_- u') \cap (Z^+[\uparrow_+ w']) \neq \varnothing$; and **(2)** $(\uparrow_+ r_2^-(u')) \cap (Z^-[\uparrow_- r_0^-(w')]) \neq \varnothing$. The worlds used in this proof are diagrammed in Figure 17.

Let $a_2 \in \uparrow_- x' \cap (Z_0^+[\uparrow_+ w'])$, where $a_0 \in \uparrow_+ w'$ such that $a_0\, Z_0^+\, a_2$. Let $b_2 \in \uparrow_- u' \cap (Z_1^+[\uparrow_+ x'])$, where $b_0 \in \uparrow_+ x'$ such that $b_0\, Z_1^+\, b_2$. By the polar reachability lemma {11} we have $x' \in \uparrow_+ a_2$ and $b_0 \in \uparrow_+ a_2$, since $a_2 \in \uparrow_- x'$ and $b_0 \in \uparrow_+ x'$. (Z_0^+, Z_0^-) is a bisimulation so by Part 1 of the frame bisimulation definition applied to $b_0 \in \uparrow_+ a_2$, we have that there exists $a_0' \in \uparrow_+ a_0$ such that $b_0\, (Z_0^+)^{-1}\, a_0'$. Again by polar reachability we have $a_0' \in \uparrow_+ w'$, since $a_0' \in \uparrow_+ a_0$ and $a_0 \in \uparrow_+ w'$. By definition of relational composition we have $a_0'\, Z^+\, b_2$, since $a_0'\, Z_0^+\, b_0$ and $b_0\, Z_1^+\, b_2$. Therefore we have $b_2 \in \uparrow_- u' \cap (Z^+[\uparrow_+ w'])$, since $b_2 \in \uparrow_- u'$ and $a_0' \in \uparrow_+ w'$.

Let $a_3 \in \uparrow_+ r_1^-(x') \cap (Z_0^-[\uparrow_- r_0^-(w')])$, where $a_1 \in \uparrow_- r_0^-(w')$ such that $a_1\, Z_0^-\, a_3$. Similarly, let $b_3 \in \uparrow_+ r_2^-(u') \cap (Z_1^-[\uparrow_- r_1^-(x')])$, where $b_1 \in \uparrow_- r_1^-(x')$ such that $b_1\, Z_1^-\, b_3$. By polar reachability {11} we have $r_1^-(x') \in \uparrow_- a_3$ and $b_1 \in \uparrow_- a_3$, since $a_3 \in \uparrow_+ r_1^-(x')$ and $b_1 \in \uparrow_- r_1^-(x')$. (Z_0^+, Z_0^-) is a bisimulation and we have $a_1\, Z_0^-\, a_3$ so therefore by Part 1 of the frame bisimulation definition applied to $b_1 \in \uparrow_- a_3$, we have that there exists $a_1' \in \uparrow_- a_1$ such that $b_1\, (Z_0^-)^{-1}\, a_1'$. We have $a_1' \in \uparrow_- r_0^-(w')$ by polar reachability, since $a_1' \in \uparrow_- a_1$ and $a_1 \in \uparrow_- r_0^-(w')$. By the definition of relation composition we have $a_1'\, Z^-\, b_3$, since $a_1'\, Z_0^-\, b_1$ and $b_1\, Z_1^-\, b_3$. Therefore we have $b_3 \in Z^-[\uparrow_- r_0^-(w')]$, since $a_1' \in \uparrow_- r_0^-(w')$ and $a_1'\, Z^-\, b_3$. Thus we conclude that $b_3 \in \uparrow_+ r_2^-(u') \cap (Z^-[\uparrow_- r_0^-(w')])$, since $b_3 \in \uparrow_+ r_2^-(u')$. □

Lemma 48. *Let F_0, F_1 and F_2 be **DCInt** frames. If the pair (Z_0^+, Z_0^-) is a bisimulation between F_0 and F_1, and the pair (Z_1^+, Z_1^-) is a bisimulation between F_1 and F_2, then the pair $((Z_0^+;Z_1^+), (Z_0^-;Z_1^-))$ is a bisimulation between F_0 and F_2.*

Proof. Define $Z^p = Z_0^p;Z_1^p$ for each polarity p. Let q be a polarity, and suppose $w\,Z^q\,u$. We will only show that the "back" condition holds, since showing the "forth" condition is similar but easier. By definition of relation composition, there exists $x \in F_1$ such that $w\,Z_0^q\,x$ and $x\,Z_1^q\,u$. Let $G_0 = F_2$ and $G_2 = F_0$, and for each polarity p define $B_0^p = (Z_1^p)^{-1}$ and $B_1^p = (Z_0^p)^{-1}$. Therefore we have $u\,B_0^q\,x$ and $x\,B_1^q\,w$. By Lemma 46, both the pairs (B_0^+, B_0^-) and (B_1^+, B_1^-) are bisimulations. This means that u is similar to x with respect to $(q, G_0, F_1, B_0^+, B_0^-)$, and x is similar to w with respect to $(q, F_1, G_2, B_1^+, B_1^-)$. By these facts together with Lemma 47, we have that u is similar to w with respect to $(q, G_0, G_2, (B_0^+;B_1^+), (B_0^-;B_1^-))$. By definition of relation composition, for each polarity p we have $(Z^p)^{-1} = (Z_0^p;Z_1^p)^{-1} = (Z_1^p)^{-1};(Z_0^p)^{-1} = B_0^p;B_1^p$. Therefore we have the "back" condition: u is similar to w with respect to $(q, F_2, F_0, (Z^+)^{-1}, (Z^-)^{-1})$. □

Finally, the bisimulation composition theorem {49} is a corollary of Lemma 48, and simply extends that result to structure bisimulations {44}.

Theorem 49 (bisimulation composition)**.** *Let M_0, M_1, and M_2 be **DCInt** structures. If the pair (Z_0^+, Z_0^-) is a bisimulation between M_0 and M_1 and the pair (Z_1^+, Z_1^-) is a bisimulation between M_1 and M_2, then $((Z_0^+;Z_1^+), (Z_0^-;Z_1^-))$ is a bisimulation between M_0 and M_2.*

5 DCInt has the disjunction property

In this section we prove that **DCInt** has the disjunction property by showing that we can **(1)** transform any given collection of **DCInt** structures into a single new synthesized **DCInt** structure; and **(2)** that the existence of the new structure can be used to imply the disjunction property for **DCInt**. This proof method is adapted from the well known model theoretic method of proving the disjunction property for **Int** (see Section 6.4 of [9] for a complete example of this approach for **Int**). The essential idea of the **Int** method is to prove the contrapositive of the disjunction property by constructing a countermodel of a disjunction $A \vee B$ from a countermodel of A and a countermodel of B. Supposing that neither A nor B is valid, we must have a countermodel M_A of A and a countermodel M_B of B. Let w_A be the world of M_A where A is not modeled, and w_B be the world of M_B where B is not modeled. It can be shown that there exists a structure M that has both M_A and M_B embedded within it, and also has an additional fresh world $\star$ that reaches both w_A and w_B.[10] Furthermore, it can be shown that the formulas modeled at w_A in M_A are exactly those that are modeled at w_A in M, and the same for w_B with respect to M_B and M. In particular, this means that in this new structure the world w_A continues to be a countermodel of A and the world w_B continues to be a countermodel of B. Finally, the world $\star$ of M is a countermodel of $A \vee B$ because **Int** has a persistence property similar to the polar persistence property {18}, and by the contrapositive of that property both A and B are countermodeled at the world $\star$.

In Section 4 we gave a counterexample to the disjunction property for **Partial BiInt** {23}, and so this means that there must be an aspect of that semantics that causes this proof method to go wrong. The problem arises from the fact that an exclusion formula represents knowledge about backwardly reachable worlds, and so adding the fresh world $\star$ below the worlds w_A and w_B can actually cause one of those worlds to model a formula that it previously did not model. For example, let $A = \urcorner\sigma$, $B = \sigma$, and M_A and M_B be **Partial BiInt** structures {23}. Let the worlds of M_A be $W_A = \{w_A\}$ and let the worlds of M_B be $W_B = \{w_B\}$. Further, let the negative valuation of M_A be such that $M_A, w_A \nvDash^- \sigma$ and let the negative valuation of M_B be such that $M_B, w_B \vDash^- \sigma$. This means that $M_A, w_A \nvDash^+ A$ and $M_B, w_B \nvDash^+ B$. However,

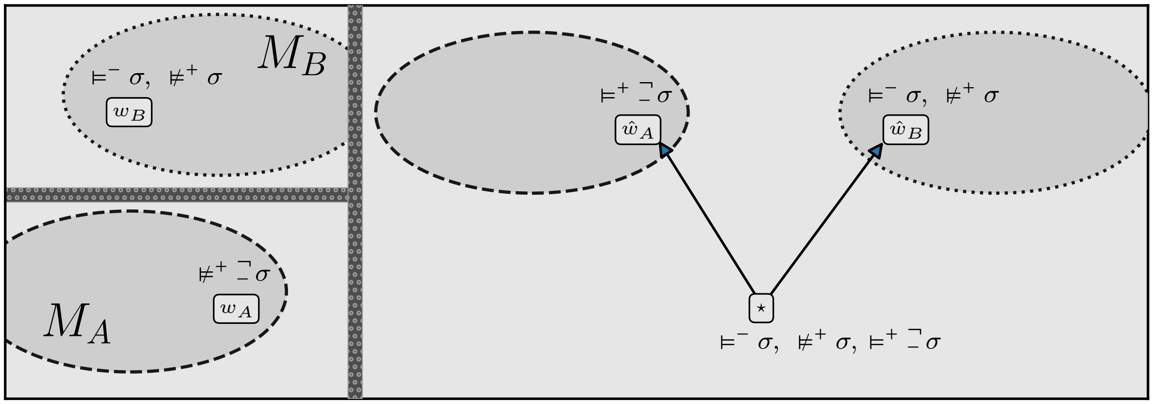
[10] Here we either assume that the pair of structures have their worlds re-labeled so that their sets of worlds are disjoint, or we assume that the embedding uses some kind of encoding to track whether a world came from M_A or M_B.

applying the fusion construction that is used for **Int** produces the structure M shown in Figure 18a, and this structure does not countermodel $A \wedge_- B$. Let $\hat{w}_A$ and $\hat{w}_B$ denote the versions of w_A and w_B that are embedded in M. We have $\star \in \uparrow_- \hat{w}_B$, so by persistence $\{18\}$ we must have $M, \star \vDash^- \sigma$. We also have $\star \in \uparrow_- \hat{w}_A$, so therefore $M, \hat{w}_A \vDash^+ \urcorner\sigma$. Thus by applying the fusion construction we caused $\hat{w}_A$ in M to positively model $\urcorner\sigma$ even though it countermodeled that formula in M_A. This eliminates the value of the fusion construction because it does not allow us to unconditionally combine a countermodel of A and a countermodel of B into a countermodel of their disjunction.

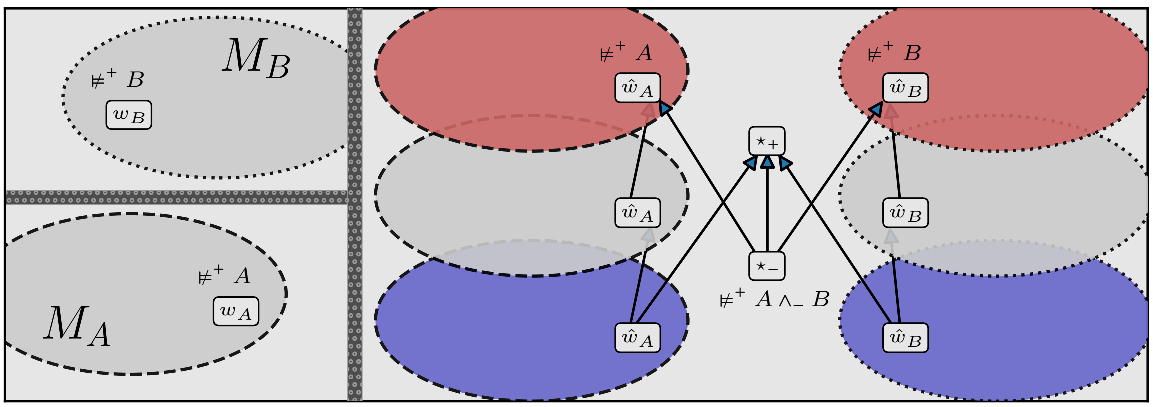
Though it is not possible to utilize the **Int** fusion construction technique in the context of **Partial BiInt**, it is possible to utilize a more complex version of the technique for **DCInt**. Essentially, a **DCInt** frame's dual counterpart components provide enough flexibility to avoid the problem that was demonstrated with **Partial BiInt**. This is quite natural, since **Partial BiInt** is defined by requiring that each dual counterpart function is the identity. The version of the fusion construction for **DCInt** is more complex because we must produce the new structure in a way that utilizes this flexibility.

We describe the general idea of the **DCInt** fusion construction as follows, where the construction transforms a collection of structures into a single new structure. Indexing by indices $i \in I$, each structure M_i of the collection will be embedded in the new structure M in such a way that there is a bisimulation between M_i and M. Furthermore, each world of the collection is embedded in the new structure three times: in the *positive level*, the *neutral level*, and the *negative level*. We replicate a world of a structure three separate times in order to faithfully represent that world in three different senses: **(1)** the version of the world on the positive level faithfully represents the world in terms of the formulas that it does and does not positively model; and **(2)** the version on the negative level faithfully represents the world in the same way but in the negative sense; and **(3)** the version in the central level faithfully represents the world in both the positive and negative sense combined. For example, the positive and central levels faithfully represent the positive sense of a world w of M_i because the bisimulation between M_i and M is defined in such a way that w will verify a formula iff both of its representations in the positive and central level verify it. The purpose of the central level is to faithfully represent a world in both senses, whereas the purpose of the positive and negative levels is intimately related to two special worlds: a fresh *negative world* denoted by $\star_-$, and a fresh *positive world* denoted by $\star_+$. Each of these worlds plays a role that is analogous to the role of $\star$ in the **Int** construction. They are fresh in the sense that they are distinct from any of the worlds that come from the collection, and they are also the only two worlds that are not part of any of the three levels. The negative world positively reaches the positive world as well as every world in the positive level (i.e. it is “below” those worlds), and the positive world negatively reaches the negative world as well as every world in the negative level (i.e. it is “above” those worlds).

For each polarity p, the **DCInt** fusion construction effectively demonstrates that a formula $A \wedge_{\overline{p}} B$ is (p)-countermodeled in the same way that the **Int** construction demonstrates that a formula $A \vee B$ is countermodeled. We will explain the general idea of why this is effective in the case of the positive polarity because that is the case that implies the disjunction property (dually, the effectiveness for the negative polarity implies the constructible falsity property). A positive countermodel for a formula of the form $A \wedge_- B$ is a **DCInt** structure M with world w such that $M, w \nvDash^+ A$ and $M, w \nvDash^+ B$; therefore, to prove the disjunction property it suffices to prove that we can combine a countermodel of A and a countermodel of B into a single countermodel of $A \wedge_- B$. Figure 18b depicts the synthesized structure M that we generate from a pair of countermodels M_A and M_B, where w_A is a world of M_A such that $M_A, w_A \nvDash^+ A$, and w_B is a world of M_B such that $M_B, w_B \nvDash^+ B$. In the context of the structure M we write $\hat{w}_A$ and $\hat{w}_B$ to distinguish the worlds w_A and w_B from their representatives in M. Later in this section we will show that there is a



(18a)



(18b)

Figure 18: The panes on the left of each subfigure show the inputs to the fusion construction: a structure where the world w_A countermodels A, and another structure where the world w_B countermodels B. The pane on the right of each subfigure diagrams the fusion construction. In the first case this is the construction that is used for proving that **Int** has the disjunction property, and in the second case this is the construction we use to prove that **DCInt** has the disjunction property. In both cases an oval with a dashed border is a copy of M_A and an oval with a dotted border is a copy of M_B. Additionally, the representatives of w_A and w_B are denoted by $\hat{w}_A$ and $\hat{w}_B$, respectively. The synthesized structure in (a) contains an embedded copy of both input structures and one fresh world denoted by $\star$. The reachability relation in the embedded structures is unchanged except for one case: this world $\star$ positively reaches $\hat{w}_A$ and $\hat{w}_B$ (and by transitivity, every world that is positively reachable from either $\hat{w}_A$ or $\hat{w}_B$). The synthesized structure in (b) contains three copies of each input structure, and so it also contains three distinct representatives for each of w_A and w_B. The positive, central, and negative levels are arranged from the top down. A world in a level of the structure that is situated lower in the diagram will positively reach its versions that are in the levels above it. In particular, the diagram shows that each hatted world positively reaches its version in the levels above. Finally, the fresh world $\star_-$ positively reaches the other fresh world $\star_+$, and also positively reaches every world in the positive level. Dually, $\star_+$ negatively reaches $\star_-$, and also negatively reaches every world in the negative level.

bisimulation between both M_A and M, and M_B and M. Further, the positive component of this bisimulation will relate the worlds w_A and w_B to their respective representatives $\hat{w}_A$ and $\hat{w}_B$ on both the central and positive levels. This means that the positive level representatives $\hat{w}_A$ and $\hat{w}_B$ will positively countermodel the formulas A and B, respectively. Now note that the negative world $\star_-$ positively reaches both of these worlds. The contrapositive of the polar persistence property {18} implies that the negative world will be a positive countermodel of both A and B, and therefore it is a positive countermodel of $A \wedge_- B$.

Before moving on to the formal details we must first explain a critical fact about the positive and negative worlds that ensures that–for example–the aforementioned **DCInt** structure M_A is bisimilar to the synthesized structure M. This is that in the structure M, the positive world is the negative counterpart of the negative world, and the negative world is the positive counterpart of the positive world. In other words, denoting the dual counterpart functions of M by r^+ and r^-, we have $r^-(\star_-) = \star_+$ and $r^+(\star_+) = \star_-$. Technically, this arrangement ensures that each world of M_A can be related by the positive component of the bisimulation to its twin in the positive level of M (and the same for the negative level with respect to the negative component of the bisimulation). For example, it ensures that the positive component of the bisimulation can relate w_A and $\hat{w}_A$ even though we have $\star_- \in \uparrow_- \hat{w}_A$ (this will be proved by the second pure world and pure fusion lemmas). From an intuitive point of view, the dual counterpart arrangement between the two fresh worlds prevents a case where the world $\hat{w}_A$ verifies an exclusion that the world w_A does not verify. Similar to the example of Figure 18a, the only way that $\hat{w}_A$ could verify an exclusion that is not verified by w_A is if the exclusion is witnessed by the world $\star_-$. However, suppose that $\star_-$ falsifies some formula ϕ due to the fact that $\hat{w}_B$ (on the positive level) falsifies ϕ (by the persistence property {18}). Also suppose that w_A does not verify $\urcorner\phi$, so that we expect $\hat{w}_A$ to also not verify that formula. The "new information" represented by ϕ is quarantined by the fact that we have $r^-(\star_-) = \star_+$: the formula ϕ is not necessarily falsified at the world $\star_+$, and therefore $\hat{w}_A$ does not necessarily verify $\urcorner\phi$. Of course, we actually wanted to conclude that $\hat{w}_A$ necessarily does not verify $\urcorner\phi$. However, this fact does follow from the full technical argument that uses bisimulations.

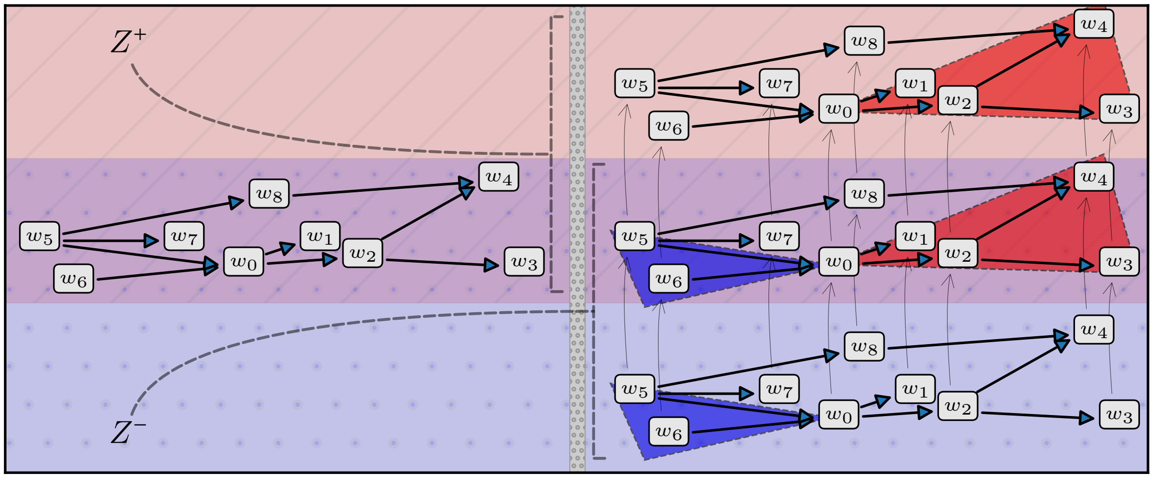
We specify the synthesized structure in two steps. We first define the *pure world extension* {51}: a three level structure that is induced by any **DCInt** structure. Figure 19a demonstrates the pure world extension of an example nine world structure. Next we define the *pure fusion* {53}: a superstructure that contains the pure world extension of each structure in a given family of structures, as well as the negative world and the positive world.

Definition 50. *Define the preorder* $\trianglelefteq$ *on the set* $\{-1, 0, +1\}$ *as the standard integer preorder restricted to the set* $\{-1, 0, +1\}$*, and define the* $[\cdot]$ *function on polarities by* $[+] = +1$ *and* $[-] = -1$.

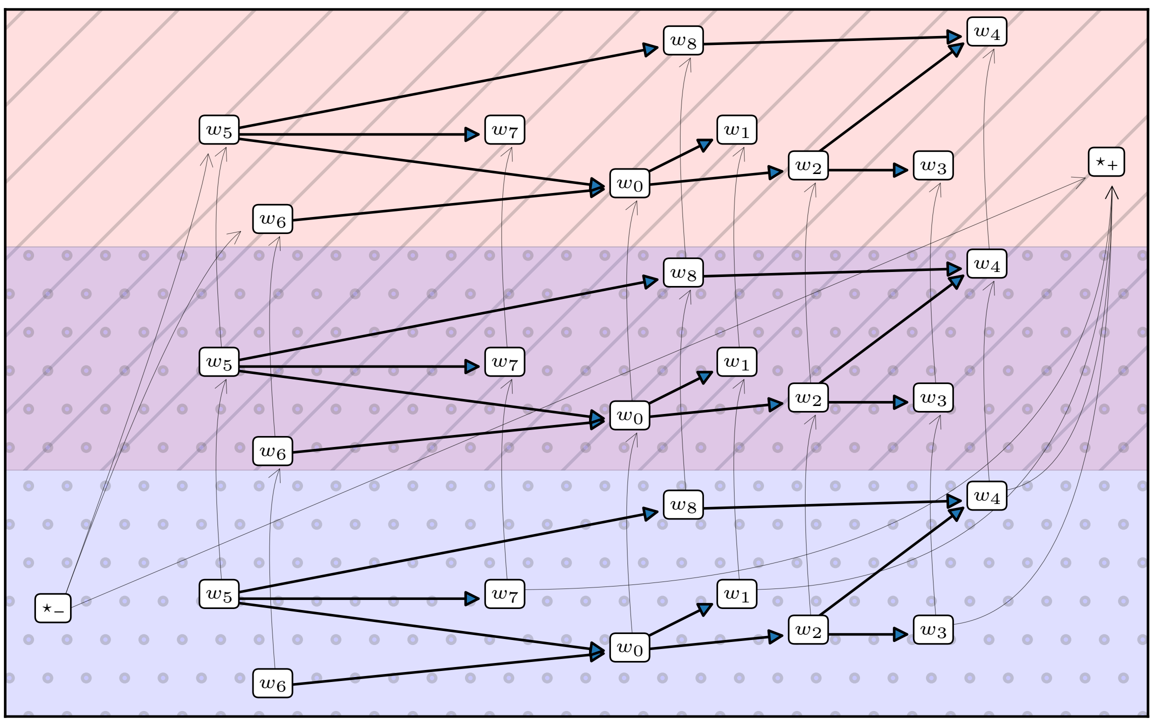
Definition 51 (pure world extension)**.** *For every* ***DCInt*** *structure* $M = \langle F, v^+, v^- \rangle$ *where* $F = \langle W, \leq, r^+, r^- \rangle$*:*

- *the* pure world extension *of* F *is the tuple* $\langle \hat{W}, \hat{\leq}, \hat{r}^+, \hat{r}^- \rangle$*, where* ***(1)*** $\hat{W} = \{-1, 0, +1\} \times W$*; and* ***(2)*** $\hat{\leq} = \{((l_0, w_0), (l_1, w_1)) \mid l_0 \trianglelefteq l_1 \text{ and } w_0 \leq w_1\}$*; and* ***(3)*** *for each polarity* p*:* $\hat{r}^p((l, w)) = (l, r^p(w))$.
- *the* pure world extension *of* M *is the tuple* $\langle \hat{F}, \hat{v}^+, \hat{v}^- \rangle$ *where* ***(1)*** $\hat{F}$ *is the pure world extension of* F*; and* ***(2)*** *for each polarity* p*,* $\hat{v}^p(\sigma) = \{(l, w) \mid l \in \{0, [p]\} \text{ and } w \in v^p(\sigma)\}$.

The pure world extension duplicates a structure across the -1, 0, and $+1$ levels. Each world (l, w) will positively reach every world (l', w') in which $l \trianglelefteq l'$ and also $w \leq w'$, and its negative reachability is defined dually. The worlds in the central level 0 have the same valuation of propositional variables as in their original structure, but the -1 and $+1$ levels only preserve the negative and positive valuations,



(19a)



(19b)

Figure 19: The pure world extension and pure fusion of an example structure, which is shown on the left side of (a). In each Figure, the lined red region covers the 0 and +1 levels, the dotted blue region covers the 0 and −1 levels, the thick arrows represent the original reachability edges, and the thin arrows represent the new edges. The Z^+ and Z^- annotations in (a) indicate the levels covered by those components of the bisimulation between the original structure and pure world extension. The solid wedges indicate the worlds to which the valuation of w_0 persists forward and backward. For example, the wedges indicate that w_0 in the central level does not negatively reach the version of w_7 on the negative level.

$$r^p(u) = \begin{cases} \star_{\overline{p}}, & u \in \{\star_{\overline{p}}, \star_p\} \\ ((i,l), r_i^p(w)), & u = ((i,l),w) \end{cases}$$

(20a) The counterpart functions

$$\begin{aligned} \leq = &\{(\star_-, u) \mid u \in L_{+1}\} \cup \{(u, \star_+) \mid u \in L_{-1}\} \\ &\cup \{(\star_-, \star_-), (\star_+, \star_+), (\star_-, \star_+)\} \\ &\cup \bigcup_{(i,l) \in J} \{(((i,l),w), ((i,l'),w')) \mid l \trianglelefteq l', \text{ and } w \leq_i w'\} \end{aligned}$$

(20b) The reachability relation, where $J = I \times \{-1, 0, +1\}$

Figure 20: Definitions for the pure fusion of $F_{i \in I}$

respectively. The set of worlds of the example shown in Figure 19a is $\{(l, w_x) \mid l \in \{-1,0,+1\},\ x \in \{0,\ldots,8\}\}$.

The pure fusion {53} is similar, except that it incorporates an arbitrary indexed family of structures and also adds the $\star_-$ and $\star_+$ worlds. For example, Figure 19b shows the pure fusion structure induced by the singleton collection comprised of the example structure from Figure 19a. This structure is nearly equivalent to the structure produced by the pure world extension, where the only difference is that the pure fusion adds $\star_-$ below the $+1$ level and adds $\star_+$ above the -1 level. Specifically, by using 0 as the index for the structure in the collection, the set of worlds of this pure fusion structure is $\{\star_-, \star_+\} \cup \{((0,l), w_x) \mid l \in \{-1,0,+1\},\ x \in \{0,\ldots,8\}\}$.

Definition 52. *Let $\mathcal{C}$ be a class of **DCInt** frames. An I indexed family of $\mathcal{C}$ structures is a non-empty set I with a family of structures $M_{i \in I}$ over Σ variables such that for every $i \in I$, the frame of M_i is in $\mathcal{C}$. For each $i \in I$ we denote the components of structure M_i by $M_i = \langle F_i, v_i^+, v_i^- \rangle$, and $F_i = \langle W_i, \leq_i, r_i^+, r_i^- \rangle$. If the class $\mathcal{C}$ is omitted, we assume the class of all **DCInt** frames.*

Definition 53 (pure fusion structure)**.** *For every I indexed family of **DCInt** structures $M_{i \in I}$:*

- *for each $l \in \{+1, 0, -1\}$ define[11] $L_l = \bigcup_{i \in I} \{(i,l)\} \times W_i$*
- *the* pure fusion *of $F_{i \in I}$ is the tuple $F = \langle W, \leq, r^+, r^- \rangle$, where **(1)** $W = \{\star_-, \star_+\} \cup L_{-1} \cup L_0 \cup L_{+1}$; and **(2)** $\leq$ is defined by Figure 20b; and **(3)** r^p is defined by Figure 20a for each polarity p*
- *the* pure fusion *of $M_{i \in I}$ is the tuple $\langle F, v^+, v^- \rangle$, where F is the pure fusion of $F_{i \in I}$ and for each polarity p, $v^p(\sigma) = \bigcup_{i \in I} \{(i,0), (i,[p])\} \times v_i^p(\sigma)$*

The first pure world lemma {54} and the first pure fusion lemma {55} each states many basic important properties that hold for the structures of Definitions 51 and 53. Both of them follow from checking definitions and the polar reachability lemma {11}, so instead of describing their proofs we will present each of their parts individually with a short explanation. Some of the parts of these lemmas will be used in the abridged proof arguments that we will provide for subsequent results, but all of the statements would be useful to a reader that works through the full proof arguments of those results.

Lemma 54 (pure world I)**.** *Let $F = \langle W, \leq, r^+, r^- \rangle$ be a frame and let $\langle \hat{W}, \hat{\lesssim}, \hat{r}^+, \hat{r}^- \rangle$ be the pure world extension of F.*

1. *for each polarity p and (l_0, w_0), $(l_1, w_1) \in \hat{W}$, we have $((l_0, w_0) \prec_p (l_1, w_1)) \in \hat{\lesssim}$ iff both $(l_0 \prec_p l_1) \in \trianglelefteq$ and $(w_0 \prec_p w_1) \in \leq$*

[11] Note that $\bigcup_{i \in I} W_i$ is a set because I is a set, and therefore L_l is also a set.

The first statement simply polarizes the definition of the reachability relation of a pure world extension structure.

2. *for each polarity p and world $(l,w) \in \hat{W}$, we have:* ***(a)*** $\uparrow_p(l,w) = \uparrow_p l \times \uparrow_p w$*; and* ***(b)*** $[p] \in \uparrow_p l$*; and* ***(c)*** $\{[p]\} \times \uparrow_p w \subseteq \uparrow_p(l,w)$

This part describes some important properties of the upset of a world, where parts (b) and (c) describe how it relates to the levels 0, -1, and $+1$. For example, a world of a level always positively reaches higher levels, so the set $\{+1\} \times \uparrow_+ w$ is a subset of $\uparrow_+(-1,w)$, $\uparrow_+(0,w)$, and $\uparrow_+(+1,w)$.

3. *for each polarity p and world $w \in W$, if $l \in \{0,[p]\}$ then we have:* ***(a)*** $\uparrow_p l \subseteq \{0,[p]\}$ *and* $\{0,[\overline{p}]\} \subseteq \uparrow_{\overline{p}} l$*; and* ***(b)*** $\uparrow_p(l,w) \subseteq \{0,[p]\} \times \uparrow_p w$*; and* ***(c)*** $\{0\} \times \uparrow_{\overline{p}} w \subseteq \uparrow_{\overline{p}}(l,w)$

Suppose for example that l is biased against the negative polarity; i.e. that we have $l \neq -1$. In this case this part says that a positive upset $\uparrow_+(l,w)$ is contained in $\{0,+1\} \times \uparrow_+ w$ (part of the central and upper levels), and that a negative upset $\uparrow_-(l,w)$ at least contains $\{0\} \times \uparrow_- w$ (part of the central level).

Lemma 55 (pure fusion I)**.** *Let $M_{i\in I}$ be an I indexed family of structures, and let $M = \langle F, v^+, v^- \rangle$ be the pure fusion of $M_{i\in I}$ with $F = \langle W, \leq, r^+, r^- \rangle$ and $J = I \times \{-1,0,+1\}$.*

1. *For every $w \in W$ and polarity p:* ***(a)*** $\star_p \prec_p w$ *iff* $w = \star_p$*; and* ***(b)*** $\uparrow_p \star_p = \{\star_p\}$*; and* ***(c)*** $\star_{\overline{p}} \prec_p w$ *iff* $w \in L_{[p]} \cup \{\star_{\overline{p}}, \star_p\}$*; and* ***(d)*** $\uparrow_p \star_{\overline{p}} = L_{[p]} \cup \{\star_{\overline{p}}, \star_p\}$

This first part establishes the reachability properties of the fresh worlds. For example, the world $\star_-$ in Figure 19b only negatively reaches itself, and positively reaches itself, $\star_+$, and every world on the positive level (i.e. $\uparrow_+ \star_- = \{\star_-, \star_+\} \cup \{((0,+1),w_x) \mid x \in \{0,\ldots,8\}\}$ and $\uparrow_- \star_- = \{\star_-\}$).

2. *For every polarity p:*
   (a) *$u \in W \setminus \{\star_+, \star_-\}$ iff there exists $(i,l) \in J$, and $w \in W_i$ such that $u = ((i,l),w)$*
   (b) *for every $(i,l),(i',l') \in J$, $w \in W_i$, and $w' \in W_{i'}$: $((i,l),w) \prec_p ((i',l'),w')$ iff* ***(1)*** $i = i'$*; and* ***(2)*** $l' \in \uparrow_p l$*; and* ***(3)*** $w' \in \uparrow_p w$
   (c) *for every $(i,l),(i',l') \in J$, $w \in W_i$, and $w' \in W_{i'}$: if both $((i,l),w) \in \uparrow_p \star_{\overline{p}}$ and $((i,l),w) \prec_p ((i',l'),w')$, then $((i',l'),w') \in \uparrow_p \star_{\overline{p}}$*
   (d) *for every $(i_1,l_1),(i_2,l_2),(i_3,l_3) \in J$, $w_1 \in W_{i_1}$, $w_2 \in W_{i_2}$, and $w_3 \in W_{i_3}$: if both $((i_1,l_1),w_1) \prec_p ((i_2,l_2),w_2)$ and $((i_2,l_2),w_2) \prec_p ((i_3,l_3),w_3)$ then $((i_1,l_1),w_1) \prec_p ((i_3,l_3),w_3)$*

This part establishes the basics of the non-fresh worlds: part (a) says that any world that is not fresh came from one of the structures of the indexed family; and part (b) says that any two of these worlds are related by reachability iff they came from the same structure and are oriented in the same way as the pure world extension; and part (c) says that–for example–every world in the positive upset of w_5 on the upper level of Figure 19b is also in the positive upset of $\star_-$; and part (d) just states that transitive reachability still holds for the worlds that came from the family of structures.

3. *For every $(i,l) \in J$, polarity p, and $w \in W_i$:*
   (a) i. $\{(i,[p])\} \times \uparrow_p w \subseteq (\{i\} \times \uparrow_p l) \times \uparrow_p w \subseteq \uparrow_p((i,l),w)$
   ii. $\uparrow_p((i,l),w) \subseteq \{\star_p\} \cup ((\{i\} \times \uparrow_p l) \times \uparrow_p w) \subseteq \{\star_p\} \cup \uparrow_p((i,l),w)$
   (b) *$l \in \{0,[p]\}$ iff* $\uparrow_p((i,l),w) = (\{i\} \times \uparrow_p l) \times \uparrow_p w$
   (c) *$l = [\overline{p}]$ iff* $\uparrow_p((i,l),w) = \{\star_p\} \cup ((\{i\} \times \uparrow_p l) \times \uparrow_p w)$

This part says that the upset $\uparrow_+((i,l),w)$ is almost equivalent to the set $(\{i\} \times \uparrow_+ l) \times \uparrow_+ w$. The equivalence is exact in the case where $l \in \{0,+1\}$, but otherwise when $l = -1$ it is equal to $\{\star_+\} \cup ((\{i\} \times \uparrow_+ l) \times \uparrow_+ w)$.

For example, in Figure 19b we have $\uparrow_+((0,-1),w_4) = \{\star_+,((0,-1),w_4),((0,0),w_4),((0,+1),w_4)\}$.

4. *For every $i \in I$, polarity p, $w \in W_i$, if $l \in \{0,[p]\}$ then:*
   (a) $\{(i,0)\} \times \uparrow_{\overline{p}} w \subseteq (\{i\} \times \uparrow_{\overline{p}} l) \times \uparrow_{\overline{p}} w \subseteq \uparrow_{\overline{p}}((i,l),w)$
   (b) $\uparrow_p((i,l),w) \subseteq \{(i,0),(i,[p])\} \times \uparrow_p w$

This part is analogous to Lemma 54.3. If l is biased against the negative polarity, then the positive upset $\uparrow_+((i,l),w)$ is contained in $\{(i,0),(i,+1)\} \times \uparrow_+ w$ (part of the central and upper levels). Additionally, the negative upset $\uparrow_-((i,l),w)$ contains $\{(i,0)\} \times \uparrow_- w$ (part of the negative level).

5. *For each polarity p, $\sigma \in \Sigma$,*
   (a) *$u \in v^p(\sigma)$ iff there exists $i \in I$, $l \in \{0,[p]\}$ and $w \in v_i^p(\sigma)$ such that $u = ((i,l),w)$*
   (b) *for each $i \in I$ we have $\{(i,0),(i,[p])\} \times v_i^p(\sigma) \subseteq v^p(\sigma)$*

Finally, this part states that a propositional variable is only modeled at a world that comes from the family of structures. Furthermore, the status of a propositional variable at a world from one of these structures is exactly determined by its status in the original structure.

The second pure world lemma {56} proves that the pure world extension {51} of a structure M is a **DCInt** structure {14}, and that there exists a bisimulation between M and its pure world extension.

Lemma 56 (pure world II)**.** *Let $M = \langle F, v^+, v^- \rangle$ be a structure over Σ variables, and let $\hat{M} = \langle \hat{F}, \hat{v}^+, \hat{v}^- \rangle$ be the pure world extension of M, where $\hat{F}$ is the pure world extension of F. Then we have*

1. *for every $\sigma \in \Sigma$ and polarity p: $\hat{v}^p(\sigma) = \{0,[p]\} \times v^p(\sigma) = \uparrow_p(\{0,[p]\} \times v^p(\sigma))$*
2. *$\hat{F}$ is a **DCInt** frame and $\hat{M}$ is a **DCInt** structure*
3. *the pair (Z^+,Z^-) is a bisimulation between M and $\hat{M}$, where (Z^+,Z^-) is defined by $Z^p = \{(w,(l,w)) \mid l \in \{0,[p]\},\ w \in M\}$ for each polarity p*

Proof. Parts (1) and (2) follow by checking definitions and using the first pure world lemma {54}. For Part (3), we only show that the second condition of the "back" property holds, since the other aspects of the proof are simpler. Define the pair (Z^+,Z^-) such that $Z^p = \{(w,(l,w)) \mid l \in \{0,[p]\}\}$, for each polarity p. Let p be a polarity, and suppose $w\,Z^p\,u$, for $w \in M$, and $u \in \hat{M}$. While reading the rest of the proof the reader will likely benefit from constructing a diagram that is similar to previous ones (such as Figure 17), as this will help to track the worlds and their relationships. By definition of Z^p we have $u = (l,w)$ with $l \in \{0,[p]\}$. For the second condition of the "back" property, we want to show that for every $u' \in \uparrow_{\overline{p}}(l,w)$ there exists $w' \in \uparrow_{\overline{p}} w$ such that $\uparrow_{\overline{p}} w' \cap ((Z^p)^{-1}[\uparrow_p u']) \neq \varnothing$ and $\uparrow_p r^{\overline{p}}(w') \cap ((Z^{\overline{p}})^{-1}[\uparrow_{\overline{p}} \hat{r}^{\overline{p}}(u')]) \neq \varnothing$. Let $u' \in \uparrow_{\overline{p}}(l,w)$. By pure world I {54.2.a} we have $\uparrow_{\overline{p}}(l,w) = \uparrow_{\overline{p}} l \times \uparrow_{\overline{p}} w$. Therefore there exists $l' \in \uparrow_{\overline{p}} l$ and $w' \in \uparrow_{\overline{p}} w$ such that $u' = (l',w')$. By the polar reachability lemma {11} this means we have $l \in \uparrow_p l'$. We also have $(l,w') \in \uparrow_p(l',w')$, because we have both $\uparrow_p(l',w') = \uparrow_p l' \times \uparrow_p w'$ and $l \in \uparrow_p l'$. By definition we have $(l,w')\,(Z^p)^{-1}\,w'$ because $l \in \{0,[p]\}$. Therefore $w' \in (Z^p)^{-1}[\uparrow_p(l',w')]$ and so we have $w' \in \uparrow_{\overline{p}} w' \cap ((Z^p)^{-1}[\uparrow_p u'])$. This satisfies the first part. For the second part, note that we have $\hat{r}^{\overline{p}}(u') = \hat{r}^{\overline{p}}((l',w')) = (l', r^{\overline{p}}(w'))$. By pure world I {54.2.c} we have $\{[\overline{p}]\} \times \uparrow_{\overline{p}} r^{\overline{p}}(w') \subseteq \uparrow_{\overline{p}}(l', r^{\overline{p}}(w'))$. That implies that we have $([\overline{p}], r^{\overline{p}}(w')) \in \uparrow_{\overline{p}}(l', r^{\overline{p}}(w'))$. We have $([\overline{p}], r^{\overline{p}}(w'))\,(Z^{\overline{p}})^{-1}\,r^{\overline{p}}(w')$ by the definition of $(Z^{\overline{p}})^{-1}$, and so therefore $r^{\overline{p}}(w') \in (Z^{\overline{p}})^{-1}[\uparrow_{\overline{p}}(l', r^{\overline{p}}(w'))]$. Thus we conclude $r^{\overline{p}}(w') \in \uparrow_p r^{\overline{p}}(w') \cap ((Z^{\overline{p}})^{-1}[\uparrow_{\overline{p}} \hat{r}^{\overline{p}}((l',w'))])$. □

Similarly, the next lemma {57} proves that there is a bisimulation between a structure and its pure fusion.

Lemma 57 (pure fusion II)**.** *Let $M_{i\in I}$ be an I indexed family of structures, and let $M = \langle F, v^+, v^- \rangle$ be the pure fusion of $M_{i\in I}$. We have* **(1)** *F is a* ***DCInt*** *frame; and* **(2)** *M is a* ***DCInt*** *structure; and* **(3)** *For every $i \in I$, the pair (Z_i^+, Z_i^-) is a bisimulation between $\hat{M}_i$ and M, where $\hat{M}_i$ is the pure world extension of M_i and (Z_i^+, Z_i^-) is defined by $Z_i^p = \{((l_0, w), ((i, l_1), w)) \mid l_0, l_1 \in \{0, [p]\},\ w \in M_i\}$ for each polarity p.*

Proof. Parts (1) and (2) follow from checking definitions and using the first fusion lemma $\{55\}$. For part (3), let $i \in I$, and $\hat{M}_i = \langle \hat{F}_i, \hat{v}_i^+, \hat{v}_i^- \rangle$ be the pure world extension of M_i, and $\hat{F}_i = \langle \hat{W}, \hat{\lesssim}, \hat{r}^+, \hat{r}^- \rangle$ be the pure world extension of F_i. By Lemma 56.2, $\hat{M}_i$ is a **DCInt** structure. We will show that the pair (Z_i^+, Z_i^-) satisfies the frame bisimulation definition $\{43\}$ with respect to $\hat{F}_i$ and F. From this and pure fusion I $\{55.5.\text{a}\}$ it is easy to show that it satisfies the structure bisimulation definition $\{44\}$ with respect to $\hat{M}_i$ and M.

Let p be a polarity, and suppose $x Z_i^p y$. By the definition of Z_i^p, there exists $l_x, l_y \in \{0, [p]\}$ and $w \in W_i$ such that $x = (l_x, w)$ and $y = ((i, l_y), w)$. We will only show that the pair satisfies the "back" property, since the "forth" property can be shown using a similar but easier argument. Furthermore, part (1) is straightforward to prove using Lemmas 55.4.b and 54.2.c, so we only explain the proof of part (2). In particular, we want to show the second part of the condition of $((i, l_y), w)$ being similar to (l_x, w) with respect to $(p, F, \hat{F}_i, (Z_i^+)^{-1}, (Z_i^-)^{-1})$. Again, while reading this proof we recommend that the reader construct a diagram (similar to Figure 17) that tracks the worlds and their relationships. Let $y' \in \uparrow_{\overline{p}}((i, l_y), w)$. We want to show that there exists $x' \in \uparrow_{\overline{p}}(l_x, w)$, such that $\uparrow_{\overline{p}} x' \cap ((Z_i^p)^{-1}[\uparrow_p y']) \neq \varnothing$, and $\uparrow_p \hat{r}_i^{\overline{p}}(x') \cap ((Z_i^{\overline{p}})^{-1}[\uparrow_{\overline{p}} r^{\overline{p}}(y')]) \neq \varnothing$. We have $\uparrow_{\overline{p}}((i, l_y), w) \subseteq \{\star_{\overline{p}}\} \cup \big((\{i\} \times \uparrow_{\overline{p}} l_y) \times \uparrow_{\overline{p}} w\big)$ by pure fusion I $\{55.3.\text{a}\}$, so we consider the cases for the form of y'.

In the first case we have $y' = \star_{\overline{p}}$. We have $l_y = [p]$ by pure fusion I $\{55.3.\text{c}\}$, since $\star_{\overline{p}} \in \uparrow_{\overline{p}}((i, l_y), w)$. We have $y = ((i, [p]), w) \in L_{[p]}$ by the pure fusion definition $\{53\}$. We have $\{0\} \times \uparrow_{\overline{p}} w \subseteq \uparrow_{\overline{p}}(l_x, w)$ by pure world I $\{54.3.\text{c}\}$, since $l_x \in \{0, [p]\}$. Therefore $(0, w) \in \uparrow_{\overline{p}}(l_x, w)$. We choose $x' = (0, w)$, and show that both parts hold. For the first part, we have $y = ((i, [p]), w) \in \uparrow_p \star_{\overline{p}}$ by polar reachability $\{11\}$, since $y' \in \uparrow_{\overline{p}} y$ by assumption. We have $((i, [p]), w)\ (Z_i^p)^{-1}\ (0, w)$ by definition of $(Z_i^p)^{-1}$. Therefore $(0, w) \in (Z_i^p)^{-1}[\uparrow_p \star_{\overline{p}}]$ and so $x' \in \uparrow_{\overline{p}} x' \cap ((Z_i^p)^{-1}[\uparrow_p y'])$. For the second part we have $r^{\overline{p}}(y') = r^{\overline{p}}(\star_{\overline{p}}) = \star_p$ by its definition in Figure 20a. Also, $((i, [\overline{p}]), r_i^{\overline{p}}(w)) \in L_{[\overline{p}]}$ by the pure fusion definition $\{53\}$. We have $\uparrow_{\overline{p}} \star_p = L_{[\overline{p}]} \cup \{\star_p, \star_{\overline{p}}\}$ by pure fusion I $\{55.1.\text{d}\}$, so therefore $((i, [\overline{p}]), r_i^{\overline{p}}(w)) \in \uparrow_{\overline{p}} \star_p$. We have $((i, [\overline{p}]), r_i^{\overline{p}}(w))\ (Z_i^{\overline{p}})^{-1}\ (0, r_i^{\overline{p}}(w))$ by definition of $(Z_i^{\overline{p}})^{-1}$, so therefore $(0, r_i^{\overline{p}}(w)) \in (Z_i^{\overline{p}})^{-1}[\uparrow_{\overline{p}} \star_p]$. We have $\hat{r}_i^{\overline{p}}(x') = \hat{r}_i^{\overline{p}}((0, w)) = (0, r_i^{\overline{p}}(w))$, so therefore $\hat{r}_i^{\overline{p}}(x') \in \uparrow_p \hat{r}_i^{\overline{p}}(x') \cap ((Z_i^{\overline{p}})^{-1}[\uparrow_{\overline{p}} r^{\overline{p}}(y')])$.

In the second case we have $y' \in (\{i\} \times \uparrow_{\overline{p}} l_y) \times \uparrow_{\overline{p}} w$. There exists $l' \in \uparrow_{\overline{p}} l_y$ and $w' \in \uparrow_{\overline{p}} w$ such that $y' = ((i, l'), w')$. We have $\{0\} \times \uparrow_{\overline{p}} w \subseteq \uparrow_{\overline{p}}(l_x, w)$ by pure world I $\{54.3.\text{c}\}$, since $l_x \in \{0, [p]\}$. Therefore $(0, w') \in \uparrow_{\overline{p}}(l_x, w)$, since $w' \in \uparrow_{\overline{p}} w$. We prove that each part holds for choice $x' = (0, w')$. For the first part we have $\{(i, [p])\} \times \uparrow_p w' \subseteq \uparrow_p((i, l'), w')$ by pure fusion I $\{55.3.\text{a.i}\}$, so therefore $((i, [p]), w') \in \uparrow_p((i, l'), w')$. We have $((i, [p]), w')\ (Z_i^p)^{-1}\ (0, w')$ by definition of $(Z_i^p)^{-1}$, so therefore $(0, w') \in (Z_i^p)^{-1}[\uparrow_p((i, l'), w')]$. Therefore $x' \in \uparrow_{\overline{p}} x' \cap ((Z_i^p)^{-1}[\uparrow_p y'])$. For the second part we have $r^{\overline{p}}(y') = r^{\overline{p}}(((i, l'), w')) = ((i, l'), r_i^{\overline{p}}(w'))$, and $\hat{r}_i^{\overline{p}}(x') = \hat{r}_i^{\overline{p}}((0, w')) = (0, r_i^{\overline{p}}(w'))$. We have $\{(i, [\overline{p}])\} \times \uparrow_{\overline{p}} r_i^{\overline{p}}(w') \subseteq \uparrow_{\overline{p}}((i, l'), r_i^{\overline{p}}(w'))$ by pure fusion I $\{55.3.\text{a.i}\}$, so therefore $((i, [\overline{p}]), r_i^{\overline{p}}(w')) \in \uparrow_{\overline{p}}((i, l'), r_i^{\overline{p}}(w'))$. We have $((i, [\overline{p}]), r_i^{\overline{p}}(w'))\ (Z_i^{\overline{p}})^{-1}\ (0, r_i^{\overline{p}}(w'))$ by definition of $(Z_i^{\overline{p}})^{-1}$, so therefore $(0, r_i^{\overline{p}}(w')) \in (Z_i^{\overline{p}})^{-1}[\uparrow_{\overline{p}}((i, l'), r_i^{\overline{p}}(w'))]$. Thus we have $\hat{r}_i^{\overline{p}}(x') \in \uparrow_p \hat{r}_i^{\overline{p}}(x') \cap ((Z_i^{\overline{p}})^{-1}[\uparrow_{\overline{p}} r^{\overline{p}}(y')])$. □

The star countermodel lemma $\{58\}$ is the heart of our proof of the disjunction property because it brings everything together for an important payoff (Lemma 59). It says that for any world from a member of the family of structures, **(1)** the positive representation of that world in the pure fusion verifies the same formulas (and the analogous statement for the negative representation, dually); and **(2)** the negative world $\star_-$ countermodels the same formulas (and the analogous dual statement for $\star_+$).

Lemma 58 (star countermodel)**.** *Let $M_{i\in I}$ be an I indexed family of **DCInt** structures, and let M be the pure fusion of $M_{i\in I}$. For each polarity p, $i \in I$, and $w \in M_i$: **(1)** $\mathbb{T}(M_i, w, p) = \mathbb{T}(M, ((i,[p]),w), p)$; and **(2)** for each formula ϕ, if $M_i, w \nvDash^p \phi$ then $M, \star_{\overline{p}} \nvDash^p \phi$.*

Proof. By the second pure fusion lemma $\{57.2\}$, M is a structure. For every $i \in I$ let $\hat{M}_i$ be the pure world extension of M_i, and define the pair (Z_i^+, Z_i^-) as in the second pure fusion lemma $\{57.3\}$. That lemma and the second pure world lemma $\{56.2\}$ imply that $\hat{M}_i$ is a structure and that the pair (Z_i^+, Z_i^-) is a bisimulation between $\hat{M}_i$ and M. For every $i \in I$, define the pair (B_i^+, B_i^-) as in the second pure world lemma $\{56.3\}$, so we have that (B_i^+, B_i^-) is a bisimulation between M_i and $\hat{M}_i$. Therefore by the bisimulation composition theorem $\{49\}$, the pair $(B_i^+;Z_i^+, B_i^-;Z_i^-)$ is a bisimulation between M_i and M for every $i \in I$. Therefore for every $i \in I$ and $w \in M_i$ we have $w\,(B_i^p;Z_i^p)\,((i,[p]),w)$.

Let p be a polarity, $i \in I$, and $w \in M_i$. Part 1 holds because the bisimulation invariance theorem $\{45\}$ implies that $\mathbb{T}(M_i, w, p) = \mathbb{T}(M, ((i,[p]),w), p)$, since $w\,(B_i^p;Z_i^p)\,((i,[p]),w)$. For the second part, let ϕ be a formula and suppose that $M_i, w \nvDash^p \phi$. By the first part, this means that $M, ((i,[p]),w) \nvDash^p \phi$. We have $((i,[p]),w) \in L_{[p]}$ by definition of $L_{[p]}$ $\{53\}$. Also, $\uparrow_p \star_{\overline{p}} = L_{[p]} \cup \{\star_{\overline{p}}, \star_p\}$ by pure fusion I $\{55.1.d\}$, so therefore we have $((i,[p]),w) \in \uparrow_p \star_{\overline{p}}$. This together with the contrapositive of the polar persistence property $\{18\}$ implies that $M, \star_{\overline{p}} \nvDash^p \phi$. □

As a corollary we have Lemma 59, which says that we can transform a collection of countermodels of formulas into a single structure that is a countermodel of the disjunction of those formulas.

Lemma 59. *Let p be a polarity, $k \geq 1$, and $I = \{i_1, \ldots, i_k\}$ such that $\phi_{i\in I}$ is a family of formulas over Σ variables. Let J be a set such that $I \subseteq J$, $M_{j\in J}$ be a family of structures, and M be the pure fusion of the family $M_{j\in J}$. If for every $i \in I$ we have $M_i \nvDash^p \phi_i$, then we have $M, \star_{\overline{p}} \nvDash^p \phi_{i_1} \wedge_{\overline{p}} \ldots \wedge_{\overline{p}} \phi_{i_k}$.*

Proof. For every $i \in I$, let $w_i \in M_i$ be such that $M_i, w_i \nvDash^p \phi_i$. By the star countermodel lemma $\{58\}$, for each $i \in I$ we have $M, \star_{\overline{p}} \nvDash^p \phi_i$. The conclusion holds by induction on k. □

Lemma 59 implies the disjunction property, and the proof arguments needed to obtain it only rely on the possibility of constructing the pure fusion structure from a family of structures. This means that any class of frames that is closed under this construction $\{60\}$ will also have the disjunction property. This argument is formalized by the class disjunction property theorem $\{61\}$, which implies that a class of frames satisfying Definition 60 induces an extension of **DCInt** that retains the disjunction property. For example, Theorem 62 shows that the class $\mathcal{C}_{\mathrm{E}}$ $\{62\}$ defines one such extension.

Definition 60 (pure fusion closed)**.** *A class $\mathcal{C}$ of **DCInt** frames is pure fusion closed iff it is not empty, and for every family $M_{i\in I}$ of $\mathcal{C}$ structures: the frame of the pure fusion of $M_{i\in I}$ is in $\mathcal{C}$.*

Theorem 61 (class disjunction property)**.** *Let $\mathcal{C}$ be a pure fusion closed class of frames. Let p be a polarity, $k \geq 1$, and $I = \{i_1, \ldots, i_k\}$ such that $\phi_{i\in I}$ is a family of formulas over Σ variables. If $\mathcal{C} \vDash^p \phi_{i_1} \wedge_{\overline{p}} \ldots \wedge_{\overline{p}} \phi_{i_k}$ then there exists $i \in I$ such that $\mathcal{C} \vDash^p \phi_i$.*

Proof. Suppose that for every $i \in I$ there exists a structure M_i with its frame in $\mathcal{C}$ such that there exists $w_i \in M_i$ with $M_i, w_i \nvDash^p \phi_i$. Let M be the pure fusion of $M_{i \in I}$. By Lemma 59, we have $M, \star_{\overline{p}} \nvDash^p \phi_{i_1} \wedge_{\overline{p}} \ldots \wedge_{\overline{p}} \phi_{i_k}$. The frame of M is a member of $\mathcal{C}$ because the class is pure fusion closed, so it demonstrates that $\phi_{i_1} \wedge_{\overline{p}} \ldots \wedge_{\overline{p}} \phi_{i_k}$ is (p)-countermodeled in $\mathcal{C}$. This proves the contrapositive. □

Theorem 62 (example pure fusion closed class)**.** *Define the class $\mathcal{C}_E$ as follows. A* ***DCInt*** *frame $\langle W, \leq, r^+, r^- \rangle$ is a member of $\mathcal{C}_E$ iff for every polarity p and world $w \in W$: if $r^p(w) \neq w$ then $r^{\overline{p}}(r^p(w)) = w$.*

The class $\mathcal{C}_E$ is pure fusion closed, and the logic defined by that class has the disjunction property.

Proof. The frame from Figure 12b satisfies the condition for membership in $\mathcal{C}_{\mathrm{E}}$, so the class is not empty. For a family $M_{i \in I}$ of $\mathcal{C}_{\mathrm{E}}$ structures, it is easy to check that each world of the pure fusion of $M_{i \in I}$ satisfies the condition that defines $\mathcal{C}_{\mathrm{E}}$. Therefore the class is pure fusion closed.

Define the logic of $\mathcal{C}_{\mathrm{E}}$ by the semantics of **DCInt**, but where every frame must be a member of $\mathcal{C}_{\mathrm{E}}$. Suppose that the formula $A \wedge_- B$ is $(+)$-valid in this semantics. This means that $A \wedge_- B$ is $(+)$-valid over $\mathcal{C}_{\mathrm{E}}$; i.e. $\mathcal{C}_{\mathrm{E}} \vDash^+ A \wedge_- B$. Since $\mathcal{C}_{\mathrm{E}}$ is pure fusion closed, Lemma 61 implies that either $\mathcal{C}_{\mathrm{E}} \vDash^+ A$ or $\mathcal{C}_{\mathrm{E}} \vDash^+ B$. Therefore A is $(+)$-valid or B is $(+)$-valid in the logic. □

The disjunction property of **DCInt** $\{64\}$ follows from the class disjunction property $\{61\}$, since the class of all **DCInt** frames is of course pure fusion closed. The second part of Theorem 64 states that **DCInt** also has the constructible falsity property. This part follows from Lemma 63, which states that (at the level of validity) a negation connective at the opposite polarity actually represents the opposite polarity.

Lemma 63. *For every formula ϕ and polarity p: ϕ is (p)-valid iff $\neg_{\overline{p}} \phi$ is $(\overline{p})$-valid.*

Proof. The left to right implication follows directly from the definitions. For the right to left implication, suppose $\phi \rightarrow_p \top_{\overline{p}}$ is $(\overline{p})$-valid, and suppose for sake of contradiction that ϕ is (p)-countermodeled. This means that there exists a structure M_0 with world $w_0 \in M_0$ such that $M_0, w_0 \nvDash^p \phi$. M_0 is trivially a family of structures indexed by $I = \{0\}$, so let $M = \langle W, \leq, r^+, r^-, v^+, v^- \rangle$ be the pure fusion structure of that family (Figure 19b depicts an example of creating a pure fusion from a singleton family of structures). This is a **DCInt** structure by the second pure fusion lemma $\{57.2\}$, and the star countermodel lemma $\{58\}$ implies that we have $M, r^p(\star_p) \nvDash^p \phi$ because we have $r^p(\star_p) = \star_{\overline{p}}$. We are assuming that $\phi \rightarrow_p \top_{\overline{p}}$ is $(\overline{p})$-valid, so therefore $M, \star_p \vDash^{\overline{p}} \phi \rightarrow_p \top_{\overline{p}}$. This implies $M, r^p(\star_p) \vDash^p \phi$ by definition of **DCInt** semantics $\{15\}$, which is a contradiction. □

Theorem 64. *For every polarity p, and formulas A and B:*

1. *(disjunction property) if $A \wedge_{\overline{p}} B$ is (p)-valid then either A is (p)-valid or B is (p)-valid*
2. *(constructible falsity property) if $\neg_{\overline{p}}(A \wedge_p B)$ is (p)-valid then either $\neg_{\overline{p}} A$ is (p)-valid or $\neg_{\overline{p}} B$ is (p)-valid*

Proof. The second pure fusion lemma $\{57.1\}$ implies that the class of all **DCInt** frames is pure fusion closed, so–just as in the proof of Theorem 62–part (1) follows directly from the class disjunction property $\{61\}$. For part (2), suppose that $\neg_{\overline{p}}(A \wedge_p B)$ is (p)-valid. Lemma 63 implies that $A \wedge_p B$ is $(\overline{p})$-valid, and so by part (1) we have that either A is $(\overline{p})$-valid or B is $(\overline{p})$-valid. Let $X \in \{A, B\}$ be a formula such that it is $(\overline{p})$-valid. By Lemma 63 again, we have that $\neg_{\overline{p}} X$ is (p)-valid. □

6 Related work

DCInt is unique because it has duality, is constructive, and is a sublogic of **BiInt**. The related works known to us only combine at most of two of those three aspects. As described in Section 2, the concept of logical duality is an important aspect of classical logic. Via the Curry-Howard correspondence, papers such as [7], [37], [10], and [11] have developed the computational view of duality in the context of classical logic. Naturally, their systems are not constructive because they allow classical reasoning. Obviously, the works that are most relevant to **DCInt** are developments related to Nelson's **N** and Rauszer's **BiInt**. The former is constructive and has duality, and the latter is the genesis of our project. The key distinction between **DCInt** and **N** is that **N** is not a sublogic of **BiInt**: the schema $(B \wedge {\sim}A) \rightarrow (B \prec A)$ is valid in **N** but not valid in **BiInt**. Furthermore, we are not aware of any other constructive conservative extension of **Int** that also defines exclusion and/or dual-intuitionistic negation in such a way that it is a sublogic of **BiInt**. Both Cornejo's logic in [5] and the logic of [26] (see also [15]) are not constructive, though they are sublogics of **BiInt** (they lack the disjunction property because they both maintain $\sigma \vee {\sim}\sigma$ as a theorem).

DCInt is similar to **N** in that the Kripke semantic interpretation of **N** also only partially interprets a given formula (it also interprets a formula as one of verified, falsified, or neither verified nor falsified). Nelson [24] and Markov [23] originally introduced **N** (independently of each other), and later both Thomason [30] and Gurevich [20] developed Kripke semantic interpretations for it that featured such partial interpretation of formulas. **N** was originally developed with a focus on the notion of verification and falsification with respect to negation instead of with respect to exclusion, and so it only extends the language of **Int** with a new unary negation connective called *strong negation* (it does not define the exclusion connective). However, Gurevich [20] identified that **N** has the duality of Figure 4b, and showed that the exclusion connective can be defined in terms of strong negation. Writing strong negation as $-$, he defines exclusion by $B \prec A \equiv -(-A \rightarrow -B)$. **N** does have the disjunction property, the constructible falsity property, and duality, but it defines exclusion as a non-modal connective. A world of a structure assigns "verified" to $B \prec A$ iff it also assigns "verified" to B and "falsified" to A, and this is more akin to the **CL** definition of exclusion from Figure 3a.

Wansing's **2Int** of [35] is another conservative extension of **Int** with duality, and it is motivated by the idea of dualizing the rules of the intuitionistic natural deduction proof system **NJ**. Its Kripke semantics is defined in a polarized style that is similar to some of our definitions for **DCInt**. Its semantic interpretation is defined on the language of **BiInt** from Figure 4a, and so it explicitly defines the meaning of the exclusion connective (as opposed to **N**, in which the meaning of exclusion is induced by its definition in terms of strong negation). It is also similar to both **N** and **DCInt** in that it only partially interprets a given formula. Furthermore, Wansing defines a translation from the language of **2Int** to the language of **Int** that faithfully preserves validity. The translation preserves disjunctions, and so as a result this implies that **2Int** has the disjunction property.[12] Though **DCInt** and **2Int** are defined on equivalent languages, they differ on some important points. First, **2Int** exhibits a kind of polarized paraconsistency because it permits some formulas to be both verified and falsified in a single world. For example, it is possible for a world of a **2Int** Kripke structure to both verify and falsify the same propositional variable. In contrast, Property 2 of the definition of **DCInt** structures $\{14\}$ ensures that this is not possible for formulas of

[12] Wansing does not state this explicitly, so we briefly justify it here. Definition 4.2 of [35] defines a translation τ that maps a **2Int** formula of the form $A \vee B$ to an **Int** formula of the form $\tau(A) \vee \tau(B)$. If $A \vee B$ is valid in **2Int**, then by Theorem 4.7 the formula $\tau(A) \vee \tau(B)$ in valid in **Int**. By the disjunction property of **Int**, either $\tau(A)$ or $\tau(B)$ is valid in **Int**. By Theorem 4.7 again, either A or B is valid in **2Int**.

DCInt (also see the polar consistence property {18}). As a result, for every propositional variable σ the formula $\neg(\sigma \prec \sigma)$ is valid in **DCInt** (also see schema (1) of Theorem 28); this formula is not valid in **2Int** because $\sigma \prec \sigma$ is satisfiable. In Section 4.3 Wansing compares **2Int** to a paraconsistent variant of **N**, and he points out that there exists a variant of **2Int** that does not exhibit the aforementioned paraconsistency. This variant is still a constructive extension of **Int** with duality, however it remains meaningfully distinct from **DCInt** because it semantically interprets exclusion as a non-modal connective. As is the case with the interpretation of exclusion in **N**, its interpretation of the verification of $B \prec A$ does not refer to non-local worlds and so it resembles the **CL** notion of exclusion. Finally, like **N** this variant also has $(B \wedge {\sim}A) \to (B \prec A)$ as a theorem, and so it is not a sublogic of **BiInt**.

In [34] Wansing investigates sixteen different ways of extending **BiInt** with the strong negation connective of **N**. The Kripke semantics of each of these logics is defined via positive and negative semantic relations, which is similar to the definitions of both **2Int** and **DCInt** Kripke semantics. In all of the sixteen logics the Kripke semantics of **BiInt** is directly embedded in the positive semantic relation. Therefore the negative semantic relation of each particular logic is the sole aspect that distinguishes it from the others. Furthermore, all the logics adopt the same interpretation as in **N** for the cases of falsified strong negation, disjunction, and conjunction. This means that each logic is actually just distinguished by its particular interpretations for falsified implication and exclusion. As a result, the logics essentially exhibit different possible hybridizations of **BiInt** and **N**, where each logic varies by demonstrating a different choice for its interpretation of the falsification of those two connectives. It is possible that some of these logics may admit a duality property, although most of the choices do not appear to exhibit the perfect symmetry that is present in **BiInt**, **N**, and **DCInt**. Furthermore, each of the sixteen interpretations of falsified exclusion and implication are meaningfully different from the interpretations in **DCInt**. Wansing argues that these logics may be viewed as constructive because when they are restricted to a certain subset of the language, they have both the disjunction property and the constructible falsity property. However, they still do not have the disjunction property with respect to their full language, which is a problem for some applications (for example, applying the logic for a type system of a programming language).

The Kripke semantics of Routley and Meyer's entailment logic of [28] (or **R**) involves a notion called a *Routley star* that is somewhat similar to the dual counterpart functions of **DCInt**. From a logical point of view, **R** differs from **DCInt** because it is more conservative with respect to implication. For instance, **R** rejects the principle of explosion and adopts principles of relevant implication. The Routley star plays a crucial role in enabling the paraconsistent character of **R** because it allows for a world to model a formula such as $A \wedge -A$, where the $-$ is the negation connective. The Routley star is formalized as a component of a Kripke frame, and it is defined as a function that maps a world w to another world w^*. Any negated formula $-A$ is then interpreted relative to w via this function, so that $w \vDash -A$ holds iff $w^* \nvDash A$ holds. The dual counterpart functions of **DCInt** are also integral to the interpretation of negated formulas such as $\urcorner A$ (from the verification perspective), but in some important ways they are playing a different role than that of the Routley star. In both cases these frame components are involved in mediating between verification and falsification, but the Routley star is in a fundamentally primary position because it directly defines which negated formulas will be modeled. Specifically, the negated formulas that are modeled at w are exactly those whose proper subformula is not modeled at w^*. In contrast, in **DCInt** the primary determination for whether a formula such as $\urcorner A$ is verified at w is based on the wide range of possible worlds in $\uparrow_{-} w$, and then secondarily depends on the dual counterpart function r^-. Furthermore, in **R** it is possible that the worlds w and w^* do not reach each other, whereas the **DCInt** frame definition {12} imposes reachability requirements on the dual counterpart functions.

7 Summary and future work

Rauszer introduced **BiInt** to create a variant of **Int** with duality. It is a conservative extension of **Int**, but yet it does not preserve the disjunction property. In some contexts this is a problem, since this property is essential for a logic to be constructive. **DCInt** is an alternative solution to Rauszer's goal because it is a conservative extension of **Int** that has duality, but also retains the disjunction property. Furthermore, it is a more appropriate solution than the other prominent extensions of **Int** because it is a sublogic of **BiInt**, which is not the case for **N** and **2Int**. It is surprising that on the one hand logics such as **N** and **2Int** retain the disjunction property while incorporating a non-modal exclusion connective that is similar to that of **CL**, and yet on the other hand a logic such as **BiInt** fails to have the disjunction property while incorporating a modal exclusion connective. We expect that future work involving **DCInt** will help to shed some light on the value of having a constructive logic that interprets exclusion as a modal connective. Finally, we conjecture that **DCInt** is also a sublogic of **N**. This fact would establish **DCInt** as the most conservative constructive extension of **Int** with duality, but we leave this to future work.

In this article we have only defined **DCInt** in terms of its Kripke semantics, so the natural next step is to develop its proof theory. We are presently working on producing a sound and complete sequent calculus system for **DCInt**, and we also plan on exploring its computational meaning via the Curry-Howard correspondence. In [4] Brunner & Carnielli define a methodology for obtaining the dual of a logic from its entailment relation, and so this approach can be employed with respect to a proof system for **DCInt** (in particular, it will be interesting to determine whether **DCInt** satisfies a condition the article defines in which a logic is said to be "self-dual").

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