Inference and Logic

Readings: Chapter 6 of Russell & Norvig.



A logic is a triple $\langle \mathcal{L}, \mathcal{S}, \mathcal{R} \rangle$ where

- \mathcal{L} , the logic's **language**, is a class of sentences described by a formal grammar.
- S, the logic's **semantics** is a formal specification of how to assign *meaning* in the "real world" to the elements of \mathcal{L} .
- \mathcal{R} , the logic's **inference system**, is a set of formal derivation *rules* over \mathcal{L} .

There are *several* logics: propositional, first-order, higher-order, modal, temporal, intuitionistic, linear, equational, non-monotonic, fuzzy, ...

We will concentrate on *propositional logic* and *first-order logic*.

Propositional Logic

Each sentence is made of

- propositional variables $(A, B, \ldots, P, Q, \ldots)$
- logical constants (True, False).
- logical connectives $(\land,\lor,\Rightarrow,\ldots)$.

Every propositional variable stands for a basic fact.

Ex: I'm hungry, Apples are red, Bill and Hillary are married.

Propositional Logic

The Language

- Each propositional variable $(A, B, \ldots, P, Q, \ldots)$ is a sentence.
- Each logical constant (**True**, **False**) is a sentence.
- If φ and ψ are sentences, all of the following are also sentences.

$$(\varphi) \quad \neg \varphi \qquad \varphi \land \psi \qquad \varphi \lor \psi \qquad \varphi \Rightarrow \psi \qquad \varphi \Leftrightarrow \psi$$

• Nothing else is a sentence.

The Language of Propositional Logic

Formally, it is the language generated by the following grammar.

• Symbols:

- Propositional variables: $A, B, \ldots, P, Q, \ldots$
- Logical constants:

True	(true)	\land (and)	\Rightarrow (implies)	\neg (not)
False	(false)	\vee (or)	\Leftrightarrow (equivalent)	

• Grammar Rules:

Propositional Logic

Ontological Commitments

Propositional Logic is about *facts* in the world that are either true or false, nothing else.

Semantics of Propositional Logic

Since each propositional variable stands for a fact about the world, its meaning ranges over the Boolean values $\{True, False\}$.

Note: Do note confuse, as the textbook does, *True*, *False*, which are values (ie semantical entities) with **True**, **False** which are logical constants (ie symbols).

Semantics of Propositional Logic

- The meaning (value) of **True** is always *True*. The meaning of **False** is always *False*.
- The meaning of the other sentences depends on the meaning of the propositional variables.
 - Base cases: Truth Tables

Р	Q	$\neg P$	$P \wedge Q$	$P \lor Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
False	False	True	False	False	True	True
False	True	True	False	True	True	False
True	False	False	False	True	False	False
True	True	False	True	True	True	True

- Non-base Cases: Given by reduction to the base cases.

Ex: the meaning of $(P \lor Q) \land R$ is the same as the meaning of $A \land R$ where A has the same meaning as $P \lor Q$.

The Meaning of Logical Connectives: A Warning

Disjunction

- $A \lor B$ is true when A or B or or both are true (inclusive or).
- $A \oplus B$ is sometimes used to mean "either A or B but not both" (exclusive or).

Implication

- $A \Rightarrow B$ does not require a causal connection between A and B. Ex: Sky-is-blue \Rightarrow Snow-is-white
- When A is false, $A \Rightarrow B$ is always true *regardless* of the value of B. *Ex:* Two-equals-four \Rightarrow Apples-are-red
- Beware of negations in implications.

Ex: $(\neg Has\text{-}blue\text{-}seal) \Rightarrow (\neg Chiquita)$

Semantics of Propositional Logic

• An assignment of Boolean values to the propositional variables of a sentence is an **interpretation** of the sentence.

Р	Н	$P \lor H$	$(P \lor H) \land \neg H$	$((P \lor H) \land \neg H) \Rightarrow P$
False	False	False	False	True
False True	True False	True True	False True	True True
True	True	True	False	True

- The semantics of Propositional logic is **compositional**:
 - The meaning of a sentence is given recursively in terms of the meaning of the sentence's components (all the way down to its propositional variables).

Semantics of Propositional Logic

The meaning of a sentence in general depends on its interpretation. Some sentences, however, have always the same meaning.

Р	Н	$P \lor H$	$(P \lor H) \land \neg H$	$((P \lor H) \land \neg H) \Rightarrow P$
False	False	False	False	True
False	True	True	False	True
True	False	True	True	True
True	True	True	False	True

A sentence is

- **satisfiable** if it is true in *some* interpretation,
- valid if it is true in *every* possible interpretation.

Entailment in Propositional Logic

Given a set Γ of sentences and a sentence φ , we write

 $\Gamma \models \varphi$

iff every interpretation that makes all sentences in Γ true makes φ also true.

 $\Gamma \models \varphi$ is read as " Γ entails φ " or " φ logically follows from Γ ."

Entailment in Propositional Logic: Examples

$$\{A, A \Rightarrow B\} \models B$$

$$\{A\} \models A \lor B$$

$$\{A, B\} \models A \land B$$

$$\{A, B\} \models A \land B$$

$$\{\} \models A \lor \neg A$$

$$\{A\} \not\models A \land B$$

$$\{A \lor \neg A\} \not\models A$$

	A	В	$A \Rightarrow B$	$A \lor B$	$A \wedge B$	$A \lor \neg A$
1.	False	False	True	False	False	True
2.	False	True	True	True	False	True
3.	True	False	False	True	False	True
4.	True	True	True	True	True	True



Entailment in Propositional Logic

Note:

- $\Gamma \models \varphi$, for all $\varphi \in \Gamma$ (inclusion property of PL)
- if $\Gamma \models \varphi$, then $\Gamma' \models \varphi$ for all $\Gamma' \supseteq \Gamma$ (monotonicity of PL)
- φ is valid iff **True** $\models \varphi$ (also written as $\models \varphi$)
- φ is unsatisfiable iff $\varphi \models \mathbf{False}$
- $\Gamma \models \varphi$ iff the set $\Gamma \cup \{\neg \varphi\}$ is unsatisfiable

Logical Equivalence

Two sentences φ_1 and φ_2 are **logically equivalent**

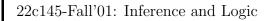
 $\varphi_1 \equiv \varphi_2$

if $\varphi_1 \models \varphi_2$ and $\varphi_2 \models \varphi_1$.

Note:

- If $\varphi_1 \equiv \varphi_2$, every interpretation of their propositional variables will assign the same Boolean value to φ_1 and φ_2 .
- Implication and equivalence $(\Rightarrow, \Leftrightarrow)$, which are *syntactical entities*, are intimately related to entailment and logical equivalence (\models, \equiv) , which are *semantical notions*:

 $\begin{array}{ll} \varphi_1 \models \varphi_2 & \text{iff} \models \varphi_1 \Rightarrow \varphi_2 \\ \varphi_1 \equiv \varphi_2 & \text{iff} \models \varphi_1 \Leftrightarrow \varphi_2 \end{array}$



Properties of Logical Connectives

• \wedge and \vee are *commutative*

$$\begin{array}{l} \varphi_1 \wedge \varphi_2 \ \equiv \ \varphi_2 \wedge \varphi_1 \\ \varphi_1 \vee \varphi_2 \ \equiv \ \varphi_2 \vee \varphi_1 \end{array}$$

• \land and \lor are *associative*

$$\begin{array}{l} \varphi_1 \wedge (\varphi_2 \wedge \varphi_3) \ \equiv \ (\varphi_1 \wedge \varphi_2) \wedge \varphi_3 \\ \varphi_1 \vee (\varphi_2 \vee \varphi_3) \ \equiv \ (\varphi_1 \vee \varphi_2) \vee \varphi_3 \end{array}$$

• \land and \lor are mutually *distributive*

$$\begin{array}{l} \varphi_1 \wedge (\varphi_2 \vee \varphi_3) \equiv (\varphi_1 \wedge \varphi_2) \vee (\varphi_1 \wedge \varphi_3) \\ \varphi_1 \vee (\varphi_2 \wedge \varphi_3) \equiv (\varphi_1 \vee \varphi_2) \wedge (\varphi_1 \vee \varphi_3) \end{array}$$

• \wedge and \vee are related by \neg (DeMorgan's Laws)

$$\neg(\varphi_1 \land \varphi_2) \equiv \neg\varphi_1 \lor \neg\varphi_2 \neg(\varphi_1 \lor \varphi_2) \equiv \neg\varphi_1 \land \neg\varphi_2$$

Properties of Logical Connectives

 $\wedge, \Rightarrow,$ and \Leftrightarrow are actually redundant:

$$\begin{array}{ll}
\varphi_1 \wedge \varphi_2 &\equiv \neg(\neg \varphi_1 \vee \neg \varphi_2) \\
\varphi_1 \Rightarrow \varphi_2 &\equiv \neg \varphi_1 \vee \varphi_2 \\
\varphi_1 \Leftrightarrow \varphi_2 &\equiv (\varphi_1 \Rightarrow \varphi_2) \wedge (\varphi_2 \Rightarrow \varphi_1)
\end{array}$$

We keep them all mainly for convenience.

Exercise

Use the truth tables to prove all the logical equivalences seen so far.

Inference Systems for Propositional Logic

- In practice, an inference system \mathcal{I} for PL is a procedure that given a set $\Gamma = \{\alpha_1, \ldots, \alpha_m\}$ of sentences and a sentence φ , may reply "yes", "no", or run forever.
- If \mathcal{I} replies positively on input (Γ, φ) , we say that Γ derives φ in $\mathcal{I}^{,1}$, and write

$\Gamma \vdash_{\mathcal{I}} \varphi$

• Intuitively, \mathcal{I} should be such that it replies "yes" on input (Γ, φ) only if φ is in fact entailed by Γ .

¹Or, \mathcal{I} derives φ from Γ , or, φ derives from Γ in \mathcal{I} .

All These Fancy Symbols!

Before we continue recall that

 $\bullet \ A \wedge B \Rightarrow C$

is a sentence, a bunch of *symbols* manipulated by an *inference system* \mathcal{I} .

 $\bullet \ A \wedge B \models C$

is a mathematical abbreviation standing for the statement: "every interpretation that makes $A \wedge B$ true, makes C also true."

 $\bullet \ A \wedge B \vdash_{\mathcal{I}} C$

is a mathematical abbreviation standing for the statement: " \mathcal{I} returns yes on input $(A \wedge B, C)$ " [C derives from $A \wedge B$ in \mathcal{I}].

In other words,

- \Rightarrow is a formal symbol of the logic, which is used by the inference system.
- \models is a shorthand we use to talk about the meaning of formal sentences.
- $\vdash_{\mathcal{I}}$ is a shorthand we use to talk about the output of the inference system \mathcal{I} .

The formal symbol \Rightarrow and the shorthands \models , $\vdash_{\mathcal{I}}$ are related as follows.

• The sentence $\varphi_1 \Rightarrow \varphi_2$ is valid (always true) if and only if $\varphi_1 \models \varphi_2$. *Example:* $A \Rightarrow (A \lor B)$ *is valid and* $A \models (A \lor B)$

	A	В	$A \lor B$	$A \Rightarrow (A \lor B)$
1.	False	False	False	True
2.	False	True	True	True
3.	True	False	True	True
4.	True	True	True	True

• A **sound** inference system can derive *only* sentences that logically follow from a given set of sentences:

if
$$\Gamma \vdash_{\mathcal{I}} \varphi$$
 then $\Gamma \models \varphi$.

• A **complete** inference system can derive *all* sentences that logically follow from a given set of sentences:

if
$$\Gamma \models \varphi$$
 then $\Gamma \vdash_{\mathcal{I}} \varphi$.

Inference in Propositional Logic

There are two (equivalent) types of inference systems of Propositional Logic:

- one based on truth tables (\mathcal{TT})
- \bullet one based on derivation rules (\mathcal{R})

Truth Tables

The inference system \mathcal{TT} is specified as follows:

 $\{\alpha_1, \ldots, \alpha_m\} \vdash_{\mathcal{TT}} \varphi \quad iff \quad all \ the \ values \ in \ the \ truth \ table \ of \\ (\alpha_1 \wedge \cdots \wedge \alpha_m) \Rightarrow \varphi \ are \ True.$

Inference by Truth Tables

• The truth-tables-based inference system is sound:

 $\begin{array}{ll} \alpha_1, \dots, \alpha_m \vdash_{\mathcal{TT}} \varphi & \text{implies truth table of } (\alpha_1 \wedge \dots \wedge \alpha_m) \Rightarrow \varphi \text{ all true} \\ & \text{implies } (\alpha_1 \wedge \dots \wedge \alpha_m) \Rightarrow \varphi \text{ is valid} \\ & \text{implies } \models (\alpha_1 \wedge \dots \wedge \alpha_m) \Rightarrow \varphi \\ & \text{implies } (\alpha_1 \wedge \dots \wedge \alpha_m) \models \varphi \\ & \text{implies } \alpha_1, \dots, \alpha_m \models \varphi \end{array}$

- It is also complete (exercise: prove it).
- Its time complexity is $O(2^n)$ where *n* is the number of propositional variables in $\alpha_1, \ldots, \alpha_m, \varphi$.
- We cannot hope to do better because a related, simpler problem (determining the satisfiability of a sentence) is NP-complete.
- However, really hard cases of propositional inference are somewhat rare.

Rule-Based Inference in Propositional Logic

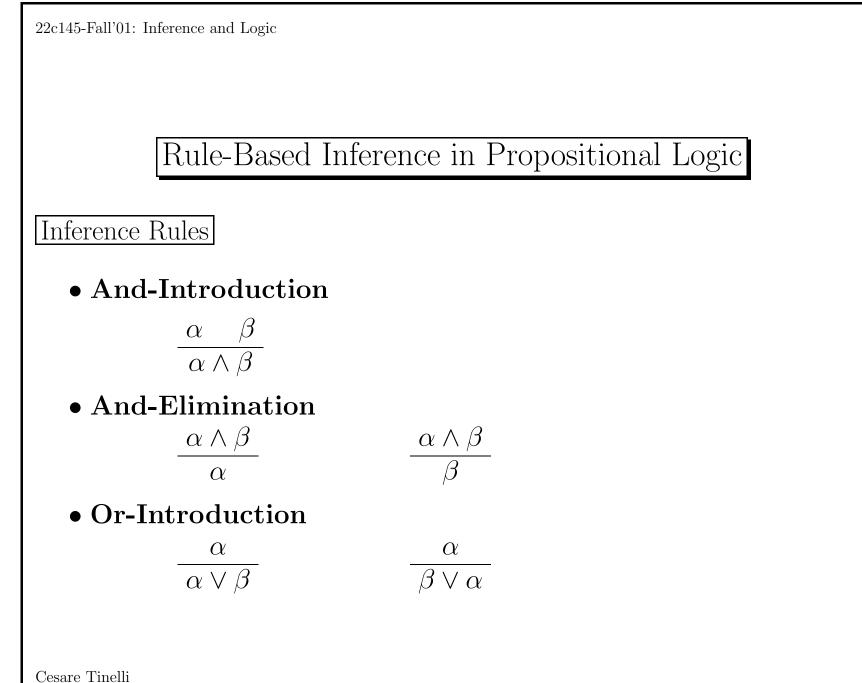
An inference system in Propositional Logic can also be specified as a set \mathcal{R} of inference (or derivation) rules.

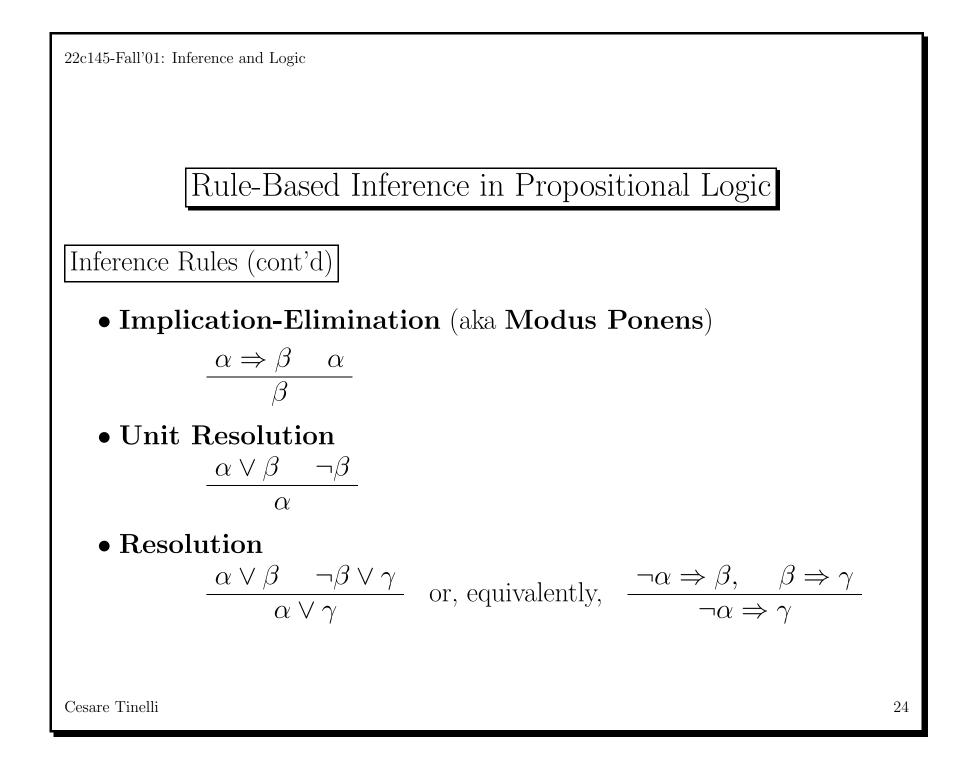
Each rule is actually a *pattern* premises/conclusion.

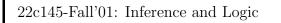
A rule applies to Γ and derives φ if

- some of the sentences in Γ match with the premises of the rule and
- φ matches with the conclusion.

A rule is **sound** it the set of its premises entails its conclusion.









Inference Rules (cont'd.)

• Double-Negation-Elimination

 $\neg \neg \alpha$

• False-Introduction $\alpha \wedge \neg \alpha$

False

• False-Elimination

 $\frac{\textbf{False}}{\beta}$

Inference by Proof

We say there is a **proof** of φ from Γ in \mathcal{R} if we can derive φ by applying the rules of \mathcal{R} repeatedly to Γ and its derived sentences.

Example: a proof of P from $\{(P \lor H) \land \neg H\}$

1. $(P \lor H) \land \neg H$	by assumption
2. $P \lor H$	by \wedge -elimination applied to (1)
$3. \neg H$	by \wedge -elimination applied to (1)
4. P	by unit resolution applied to $(2), (3)$

We can represent a proof more visually as a *proof tree*:

Example:

$$\frac{(P \lor H) \land \neg H}{P \lor H} \quad \frac{(P \lor H) \land \neg H}{\neg H}$$

$$P$$

Rule-Based Inference in Propositional Logic

More formally, there is a proof of φ from Γ in \mathcal{R} if

- 1. $\varphi \in \Gamma$ or,
- 2. there is a rule in \mathcal{R} that applies to Γ and produces φ or,
- 3. there is a proof of each $\varphi_1, \ldots, \varphi_m$ from Γ in \mathcal{R} and a rule that applies to $\{\varphi_1, \ldots, \varphi_m\}$ and produces φ .

Then, the inference system \mathcal{R} is specified as follows:

 $\Gamma \vdash_{\mathcal{R}} \varphi$ iff there is a proof of φ from Γ in \mathcal{R}

\mathcal{R} is sound because all of its rules are sound.

Example: the Resolution rule
$$\frac{\alpha \lor \beta, \neg \beta \lor \gamma}{\alpha \lor \gamma}$$

	α	β	γ	$\neg\beta$	$\alpha \vee \beta$	$\neg\beta\vee\gamma$	$\alpha \vee \gamma$
1.	False	False	False	True	False	True	False
2.	False	False	True	True	False	True	True
3.	False	True	False	False	True	False	False
4.	False	True	True	False	True	True	True
5.	True	False	False	True	True	True	True
6.	True	False	True	True	True	True	True
7.	True	True	False	False	True	False	True
8.	True	True	True	False	True	True	True

All the interpretations that make both $\alpha \lor \beta$ and $\neg \beta \lor \gamma$ true (ie, 4,5,6,8) make $\alpha \lor \gamma$ also true.

- Exercise: prove that the other inference rules are sound as well.
- Is \mathcal{R} also complete?

	The rules of \mathcal{R}				
$\frac{\alpha \beta}{\alpha \wedge \beta}$	$\frac{\alpha}{\alpha \lor \beta}$	$\frac{\alpha}{\beta \lor \alpha}$			
$\frac{\alpha \wedge \beta}{\alpha}$	$\frac{\alpha \wedge \beta}{\beta}$				
$\frac{\alpha \Rightarrow \beta \alpha}{\beta}$	$\frac{\alpha \lor \beta \neg \beta}{\alpha}$	$\frac{\alpha \lor \beta \neg \beta \lor \gamma}{\alpha \lor \gamma}$			
$\frac{\neg \neg \alpha}{\alpha}$	$\frac{\alpha \wedge \neg \alpha}{\mathbf{False}}$	$\frac{\textbf{False}}{\beta}$			