22c:196:002 Logic in Computer Science
Spring 2007

Midterm I
Selected Solutions

1 English to Propositional Logic (16 points)

1. p: Bill met Jane yesterday
   q: they went to the movies
   r: they went to the park

   $p \rightarrow (q \vee r)$

2. p: the alarm goes off
   q: there is smoke

   $(p \rightarrow q) \wedge (q \rightarrow p)$

3. p: I get sad
   q: it rains

   $q \rightarrow p$

4. p: I'll call you
   q: I get your message first

   $\neg q \rightarrow p$

   Equivalently: $\neg p \rightarrow q$

5. p: I'll call you
   q: I'm done
   r: I'll see you there

   $(\neg q \rightarrow p) \wedge (q \rightarrow r)$

6. p: you hurry up
   q: you will lose the plane

   $\neg p \rightarrow q$

   Equivalently,

   $p \vee q$

7. p: you suffer pain
   q: you make some gain.

   $\neg p \rightarrow \neg q$

8. p: Jane wants a black and white cat

   p

2 Natural Deduction

1. Sequent: $\vdash ((p \to q) \land (p \to r)) \to (p \to (q \land r))$

   Proof:

1	$(p \to q) \land (p \to r)$	assumption
2	$p \to q$	$\land\mathcal{E}$ 1
3	$p \to r$	$\land\mathcal{E}$ 1
4	p	assumption
5	q	$\to\mathcal{E}$ $4, 2$
6	r	$\to\mathcal{E}$ $4, 3$
7	$p \to (q \land r)$	$\to\mathcal{I}$ 4–6
8	$((p \to q) \land (p \to r)) \to (p \to (q \land r))$	$\to\mathcal{I}$ 1–7

2. Sequent: $(p \land q) \to r,\ r \to s,\ q \land \neg s\ \vdash\ \neg p$

   Proof:

1	$(p \land q) \to r$	premise
2	$r \to s$	premise
3	$q \land \neg s$	premise
4	q	$\land\mathcal{E}$ 3
5	p	assumption
6	$p \land q$	$\land\mathcal{I}$ $5, 4$
7	r	$\to\mathcal{E}$ $6, 1$
8	s	$\to\mathcal{E}$ $7, 2$
9	$\neg s$	$\land\mathcal{E}$ 3
10	$\bot$	$\neg\mathcal{E}$ $8, 9$
11	$\neg p$	$\neg\mathcal{I}$ 5–10

   Alternative Proof:

1	$(p \wedge q) \rightarrow r$	premise
2	$r \rightarrow s$	premise
3	$q \wedge \neg s$	premise
4	$\neg s$	$\wedge\mathcal{E}$ 3
5	$\neg r$	MT 2, 4
6	$\neg(p \wedge q)$	MT 1, 5
7	p	assumption
8	q	$\wedge\mathcal{E}$ 3
9	$p \wedge q$	$\wedge\mathcal{I}$ 7, 8
10	$\bot$	$\neg\mathcal{E}$ 6, 9
11	$\neg p$	$\neg\mathcal{I}$ 7–10

3. Sequent: $p \rightarrow q \vdash (\neg p \rightarrow q) \rightarrow q$

Proof:

1	$p \rightarrow q$	premise
2	$\neg p \rightarrow q$	assumption
3	$\neg q$	assumption
4	$\neg p$	MT 1, 3
5	q	$\rightarrow\mathcal{E}$ 4, 2
6	$\bot$	$\rightarrow\mathcal{E}$ 3, 5
7	q	PBC 3–6
8	$(\neg p \rightarrow q) \rightarrow q$	$\rightarrow\mathcal{I}$ 2–7

Alternative Proof:

1	$p \rightarrow q$	premise
2	$\neg p \rightarrow q$	assumption
3	$p \vee \neg p$	LEM
4	p	assumption
5	q	$\rightarrow\mathcal{E}$ 4, 1
6	$\neg p$	assumption
7	q	$\rightarrow\mathcal{E}$ 6, 2
8	q	$\vee\mathcal{E}$ 3, 4–5, 6–7
9	$(\neg p \rightarrow q) \rightarrow q$	$\rightarrow\mathcal{I}$ 2–8

3 CNF conversion

4 DPLL

5 Entailment

MT)

Let I be an arbitrary interpretation that satisfies the premises $\alpha \to \beta$ and $\neg\beta$ of the rule. We show that I satisfies the conclusion $\neg\alpha$.

Since I satisfies $\neg\beta$ it must make β false. Then, the only way for I to satisfy $\alpha \to \beta$ is to make α false as well. But this implies that I satisfies $\neg\alpha$.

LEM)

Let I be an arbitrary interpretation that satisfies all the premises of the rule. We show that I satisfies the conclusion $\alpha \vee \neg\alpha$.

Since there are no premises, I is any interpretation of the variables of α. By the semantics of propositional logic, I makes either α true or $\neg\alpha$ true. In either case then, I will satisfy the disjunction $\alpha \vee \neg\alpha$.

6 Equivalence procedure

Let φ_1 and φ_2 be the formulas in question.

One procedure is to build the truth table of φ_1 and φ_2 and then check that for each line of the table φ_1 and φ_2 have exactly the same truth-values.

Alternative procedures are to check that the formula $(\varphi_1 \to \varphi_2) \wedge (\varphi_2 \to \varphi_1)$ is valid, or that its negation is unsatisfiable. The first check can be done with the CNF conversion procedure. The second can be done by translating the negated formula into CNF and giving it to the DPLL procedure.

7 Extra credit

The proposed procedure will not work because it does not terminate for all inputs φ. In fact, let φ be a satisfiable but not valid formula (such as, for instance, $p \wedge \neg q$, or even just p). The negation of such formulas is also satisfiable. That means that neither copy of P, the one with φ and the one that with $\neg\varphi$, are guaranteed to terminate.