22C:31 Homework 5

Due in class on Friday, Dec 18th

1. Show that $\leq_P$ is transitive, i.e., if $X_1 \leq_P X_2$ and $X_2 \leq_P X_3$ then $X_1 \leq_P X_3$.

2. Even though 3SAT is NP-complete, 2SAT is in $\mathcal{P}$. This problem will guide you to a polynomial-time algorithm for 2SAT.

   (a) Consider the following instance of 2SAT with 5 clauses:

   $$C_1 = (\overline{x_1} \vee x_2), C_2 = (\overline{x_1} \vee \overline{x_3}), C_3 = (x_1 \vee \overline{x_2}), C_4 = (x_3 \vee \overline{x_4}), C_5 = (\overline{x_1} \vee \overline{x_4}).$$

   Can you find *all* the truth assignments to the 4 variables x_1, x_2, x_3, x_4 that will satisfy all of the clauses above?

   (b) Give an instance of 2SAT with 4 variables and 5 clauses that has no satisfing truth assignment.

   (c) There is a way to transform the problem of finding whether a given instance I of 2SAT is satisfiable or not into a problem involving searching a directed graph. Let $X = \{x_1, x_2, \ldots, x_n\}$ be the boolean variables in I. Construct a directed graph G_I whose vertices are x_i and $\overline{x}_i$ for all i. So G_I has $2n$ edges. For every clause $(a \vee b)$ in I add directed edges from the literal $\overline{a}$ to the literal b and from the literal $\overline{b}$ to the literal a. The edge $(\overline{a}, b)$ indicates that if literal $\overline{a}$ were set to true then the literal b would have to be set to true in order to make the clause $(a \vee b)$ true. If there are m clauses in I, then G_I would have $2m$ edges. Follow the construction described here to construct the directed graph G_I for the instance I you came up with in (b).

   (d) Show that the directed graph you constructed for the instance in (b) has a strongly connected component containing some literal x and its negation $\overline{x}$. Then show that for any instance I of 2SAT, if G_I has a strongly connected component containing both x and $\overline{x}$ for some literal x, then I does not have a satisfying truth assignment.

   (e) Then show the converse of the above claim: namely, that if none of G_I's strongly connected components contain both a literal and its negation, then the instance I must be satisable.
   Hint: Assign values to the variables as follows: repeatedly pick a sink strongly connected component of G_I . Assign value true to all literals in the sink, assign false to their negations, and delete all of these. Show that this ends up discovering a satisfying assignment.

   (f) Finish up the problem by describing a polynomial time algorithm for 2SAT. You don't have to prove that the running time of your algorithm is polynomial in the input size.

   If you are not sure what strongly connected components in a directed graph are, look up Section 3.5.

3. Problem 1 in Chapter 8. Page 505.

4. Problem 2 in Chapter 8. Page 505.
   Hint: Reduce from `MIS-DEC`.

5. Problem 3 in Chapter 8. Pages 505-506.

6. Problem 4 in Chapter 8. Page 506.

7. Problem 5 in Chapter 8. Pages 506-507.