22C:296 Seminar on Randomization

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High Girth and Large Chromatic Number

Examples of the *first moment* method: The first example we discussed related to k-SAT. Here we discuss a second example. We know that for any graph G, $\chi(G) \geq \omega(G)$. There are example graphs G for which $\chi(G) > \omega(G)$. An example is an odd cycle of length of ≥ 5. This gap can be arbitrary as shown in the following theorem.

Theorem 1 *For every natural number k, there is a graph G_k such that $\chi(G) = k$ and $\omega(G) = 2$.*

Another way of stating the theorem is : For any natural number k, there exists a triangle free garph G_k with $\chi(G) = k$ This proof is by construction and does not use the probabilistic method.
Proof:

1. For $k = 1, 2, 3$ the claim is trivial.
2. For $k > 3$ we construct G_k from G_{k+1} as follows:

Let $V(G_{k-1}) = \{v_1, v_2, ..., v_t\}$. Here t depends on the value of k. Let $V(G_k) = \{u_1, u_2, ..., u_t\} \cup \{w\}$. The edges of G_k are obtained by taking the edges of G_{k-1} and adding edges between:

1. each u_i and each neighbour of v_i.
2. w and each u_i.

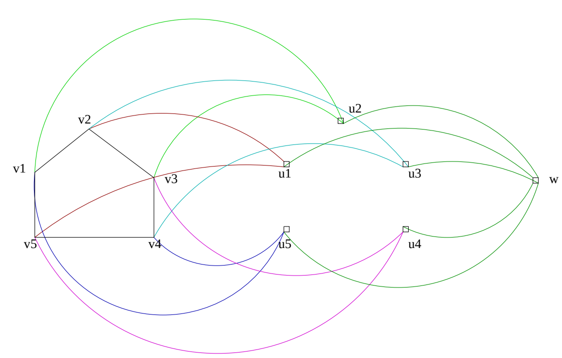

Figure 1:

Lemma 2 $\omega(G) = 2$

Proof: Let G_{k-1} be a triangle free graph. From the way we have constructed G_k we can clearly see that G_k is triangle free. $\square$

Lemma 3 $\chi(G) = k$

Proof: Since $\chi(G_{k-1}) = (k-1)$(by induction) we know that G_k needs atleast $(k-1)$ colors. We will now show that $(k-1)$ colors do not suffice for G_k. To obtain a contradiction suppose there is a $(k-1)$ coloring of G_k, w.l.o.g. let $color(w) = (k-1)$. Hence vertices in $\{u_1, u_2, ..., u_t\}$ are colored with colors in $\{1, 2, ..., k-2\}$. Consider vertex v_i that is colored $(k-1)$. If such a vertex v_i does not exist, that would mean that G_{k-1} is $(k-2)$-colorable - a contradiction. Let $color(u_i) = j$, for some j, $1 \leq j \leq k-2$, then the neighbours of v_i cannot be colored j, because they are adjacent to u_i as well. Change the color of v_i from $(k-1)$ to j. Repeat until there are no vertices colored $(k-1)$. This implies there is $(k-2)$ coloring of G_{k-1} which is a contradiction. Hence $\chi(G_k) \geq k$. A k-coloring of G_k is obtained by starting with a $(k-1)$ coloring of G_{k-1} and assigning to each u_i the color of v_i, then assign k as the color of w. $\square$

The two questions that arise are:

1. How does $|V(G_k)|$ grow?
2. What is the smallest value n such that there is a triangle free-graph with n vertices and chromatic number k?

$\square$

Erdos in 1959 proved the following thoerem.

Theorem 4 *For any pair of natural numbers g and l there exists a graph $G_{g,l}$ such that the girth of $G_{g,l} > g$ and the chromatic number $\chi(G_{g,l}) > l$*

The *girth* of a graph G is the size of the smallest cycle in G.

Proof: Choose $\theta < \frac{1}{l}$. Let $p = n^{\theta - 1}$. Let G be a random graph with n vertices, whose edges are chosen independently with probability p. (Such a graph is said to be a random graph from the model $G(n, p)$). Let X be the random variable denoting the number of cycles of length $\leq l$. We want to calculate $E[X]$. Let X_i denote the number of cycles in G of length i. So, $X = \sum_{i=3}^{l} X_i$ and $E[X] = \sum_{i=3}^{l} E[X_i]$ (by the linearity of expectation).

$$E[X_i] = \frac{\binom{n}{i} \cdot p^i \cdot i!}{2 \cdot i}$$

$$E[X] = \sum_{i=3}^{l} \frac{n!}{(n-i)!} \cdot \frac{p^i}{2 \cdot i} =$$

$$\sum_{i=3}^{l} \frac{n(n-1)...(n-(i-1))}{2 \cdot i} \cdot p^i$$

$$\leq \sum_{i=3}^{l} \frac{n^i \cdot p^i}{2 \cdot i}$$

$$= \sum_{i=3}^{l} \frac{n^i \cdot (n^{\theta-1})^i}{2 \cdot i} = \sum_{i=3}^{l} \frac{n^{\theta \cdot i}}{2 \cdot i}$$

$$\leq n^{\theta \cdot l} \cdot \sum_{i=3}^{l} \frac{1}{2i} = O(n^{\theta \cdot l} \cdot log(n)) = o(n)$$

The last equality follows from the fact that

$$\theta < \frac{1}{l} \Rightarrow \theta \cdot l < 1$$

Thus the expected length of cycles with $\theta < \frac{1}{l}$ and $p = n^{\theta-1}$ is sublinear. We now use Markov's Inequality:

$$Prob[X \geq \frac{n}{2}] \leq \frac{E[X]}{\frac{n}{2}} = \frac{o(n)}{\frac{n}{2}} = o(1)$$

Let $\alpha(G)$ denote the *independence number* of G. That is, the size of the largest independent set in G. Then

$$\text{Prob}[\alpha(G) \geq x] = \text{Prob}[\text{there exists an independent set of size} = x] \leq \binom{n}{x} \cdot (1-p)^{\binom{x}{2}}.$$

Set $x = \lceil \frac{3}{p} \cdot ln(n) \rceil$. For this value of x we can show that

$\binom{n}{x} \cdot (1-p)^{\binom{x}{2}} = O(1) \Rightarrow Prob[\alpha(G) \geq x] = o(1)$

Choose n large enough so that $\text{Prob}[X \geq \frac{n}{2}] < \frac{1}{2}$ and $\text{Prob}[\alpha(G) \geq \lceil \frac{3}{p} \cdot ln(n) \rceil] < \frac{1}{2}$. This implies that

$$\text{Prob}[X \geq \frac{n}{2} \text{ or } \alpha(G) \geq \lceil \frac{3}{p} \cdot ln(n) \rceil] < 1.$$

From this it follows that

$$\text{Prob}[X < \frac{n}{2} \text{ and } \alpha(G) < \lceil \frac{3}{p} \cdot ln(n) \rceil] > 0$$

This shows that there exists a particular graph G with n vertices such that:

1. The number of cycles in G of length $\leq g$ is $< \frac{n}{2}$.
2. $\alpha(G) < \lceil \frac{3}{p} \cdot ln(n) \rceil$

Now we use the *alteration technique*. We construct a graph G^* from G by deleting at most one vertex from each cycle of $length \leq g$ thereby destroying all cycles of $length \leq g$. Hence, $girth(G^*) > g$. Also $|V(G^*)| > \frac{n}{2}$ and $\alpha(G^*) < \lceil \frac{3}{p} \cdot ln(n) \rceil$.
Finally since $\chi(G^*) \cdot \alpha(G^*) \geq |V(G^*)| \Rightarrow \chi(G^*) \geq \frac{|V(G^*)|}{\alpha(G^*)}$

$$\Rightarrow \chi(G^*) \geq \frac{\frac{n}{2}}{\lceil \frac{3}{p} \cdot ln(n) \rceil} = \frac{n^\theta}{6 \cdot ln(n)}$$

This function grows as n does. Therefore by choosing n large enough we can make $\chi(G^*) > k$.
□