22C:253 Seminar on Randomization 22C:296

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Edge-Disjoint Cycles in k-regular Graphs

We will prove the following result :

In any k-regular graph with no parallel edges, a collection of atleast $\Omega(k^2)$ edge-disjoint cycles exist.

First, we will look at a result that proves a linear lower bound and then we will look at a proof that uses the semi-random method, or Rodl's nibble.

Theorem 1 *If G is a k-regular digraph with no parallel edges, then G has* $\geq \frac{5k}{2} - 2$ *edge-disjoint cycles.*

We prove a slightly more general result for an eulerian digraph with minimum vertex degree is $\geq$ k.

Let x be a vertex in G with degree k.

We can choose k edge-disjoint cycles passing through x.

Let $\{C_1, \cdots, C_k\}$ be k edge-disjoint cycles passing through x with *minimum* total length.

Let S denote the set of all edges in all k cycles. Let D denote the subgraph of G induced by these cycles.

Let H be the subgraph of D obtained by deleting edges incident on x.

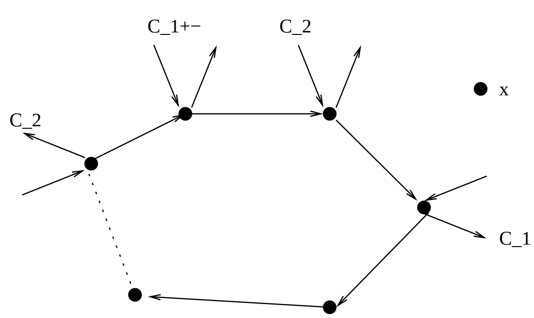

Figure 1: The figure shows how we can get smaller cycles

Lemma 2 *H is acyclic.*

Proof : Suppose H contains a cycle, edges are used by some C_i and the vertex x is not invovled. Figure **??** shows how we can get cycles of smaller total length contradicting the assumtion about the smallest total length edge-disjoint cycles. If cycle C_1 enters the cycle at the point shown in the figure, and cycle C_2 enters at the point shown in the figure, and they exit along the edges shown, and so on. Now, we can create a smaller edge-disjoint cycle for C_1, leaving by the out-edge adjacent to the same vertex that C_1 enters the cycle, and so on for each cycle. □

Since H is acyclic, it induces a partial order $\prec$ on the vertices of H.

Lemma 3 *If $u \prec v$, then G has no (u,v) edge outside of S.*

Proof : Suppose there is an edge (u,v) in $G - S$, and $u \prec v$, then Figure **??** shows how we can get a smaller cycle, contradicting our assumption about the smallest total length cycles. In the figure, the dotted edge belongs to $G - S$. □

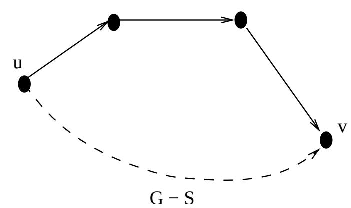

Figure 2: The figure shows how we can get smaller cycles if there was a (u, v) edge in $G - S$

Consider a minimal element y of $\prec$, in S. We see that exactly one cycle in $\{C_1, \cdots, C_k\}$ passes through y. Hence, we can get another $k - 1$ edge-disjoint cycles that pass through y.

continued in the next lecture....