22C:296 Seminar on Randomization
Lecture 1: The Probabilistic Method

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The main topics of this course are:

1. Probabilistic Method
2. Random Graphs
3. Random Walks
4. Approximate Counting
5. Randomized Rounding(tentative)

Two features of this course will be (i) the results we discuss are not always algorithmic and (ii) the tools we use from probability theory are quite elementary, but lead to powerful consequences.

1 The Probabilistic method

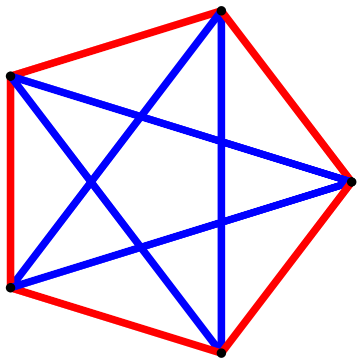
Our goal is to show the existence of a structure with a certain property. The basic idea of the *probabilistic method* is that we a probability space and show that the desired property holds in the the probability space with positive probability. The main tools we use will be:

1. First moment method
2. Second moment method.
3. Lovasz local lemma
4. Chenoff-Hoeffding bounds.
5. Martingales and Azuma's inequality.

We start with the classic example of the use of the probabilistic method.

Example 1 : Ramsey Numbers
Definition: The *Ramsey Number* $R(k, \ell)$ is the smallest integer n such that in *any* edge-coloring (not necessarily proper) of K_n with 2 colors, red and blue, there is either a red K_k or a blue K_ℓ.

For example, let us try to figure out $R(3, 3)$. Here is an edge-coloring of K_5 with colors red and blue that does not contain a monochromatic K_3, implying that $R(3, 3) > 5$.



We now show that $R(3,3) \leq 6$. Consider K_6, with the vertices labeled $1, 2, \ldots, 6$. Consider vertex 1. Since there are 5 edges incident on 1, at least 3 of these have the same color say red. Without loss of generality, suppose that edges $\{1,2\}$, $\{1,3\}$, and $\{1,4\}$ are colored red. To avoid a red triangle, edges $\{2,3\}$, $\{3,4\}$ and $\{1,4\}$ all have to be blue. This implies a blue triangle. Hence, any edge-coloring of K_6 with 2 colors contains a monochromatic K_3, and hence $R(3,3) = 6$.

It is not obvious that $R(k,\ell)$ is finite for every k and ℓ. Frank Ramsey (in 1930) showed the finiteness of $R(k,\ell)$. We are interested in bounds on $R(k,\ell)$. Our first example of the probabilistic method shows a lower bound on $R(k,k)$.

Theorem 1 *If* $\binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1$ *then* $R(k,k) > n$. *This implies that* $R(k,k) > 2^{k/2}$.

Proof: (Erdös, 1947) Suppose that n and k are natural numbers such that $\binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1$. Randomly color the edges of K_n red or blue. Specifically, pick each edge and independently assign it either red or blue with equal probability. We now want to show

$$\text{Prob[there is a coloring that does not contain a monochromatic } K_k] > 0.$$

To show this we show that

$$\text{Prob[every coloring contains a monochromatic } K_k] < 1.$$

Fix a subset S of the vertices of size k and let A_S denote the event:

$$A_S \equiv S \text{ induces a monochromatic subgraph.}$$

Then,

$$\text{Prob}[A_S] = \left(\frac{1}{2}\right)^{\binom{k}{2}}.$$

We are interested in computing the probability that there exists a subset S of vertices of size k such that S induces a monochromatic subgraph.

$$\text{Prob}[\bigcup_S A_S] \leq \sum_S \text{Prob}[A_S] = \binom{n}{k} \cdot 2^{1-\binom{k}{2}}.$$

So if $\binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1$ then $R(k,k) > n$. $\square$