# 22C:296 Seminar on Randomization
# Homework 2

Due: December 8

1. At the end of the Alon-McDiarmid-Molloy paper (*Journal of Graph Theory, 22(3), 231–237, 1996*) on edge disjoint cycles in regular directed graphs, there is the following theorem.

   > If $G$ is a $k$-regular directed graph with no parallel edges and with $k \geq 2$, then $G$ contains a collection of at least $k^2/8\ln(k)$ edge disjoint cycles.

   Note that this is weaker than the main result of the paper that proves a $\Omega(k^2)$ bound on the number of edge disjoint cycles. This weaker result can be obtained by a more straightforward application of the Lovasz Local Lemma to repeatedly find vertex disjoint cycles as before, but without the iterated splitting procedure. Prove this weaker result.

2. Consider a *1-dimensional random walk with a reflecting barrier*, which is defined as follows. For each natural number $i$, there is a state $i$. At state 0, with probability 1 the walk will move to state 1. At every other state $i > 0$, the walk will move to state $i + 1$ with probability $\rho$ and to state $i - 1$ with probability $1 - \rho$. Prove the following for the resulting Markov chain:

   (a) For $\rho > 1/2$, each state is transient.

   (b) For $\rho = 1/2$, each state is null-persistent.

   (c) For $\rho < 1/2$, each state is non-null persistent.

3. Let $G$ be a 3-colorable graph. Consider the following algorithm for coloring the vertices of $G$ with 2 colors such that no triangle of $G$ is monochromatic. The algorithm begins with an arbitrary 2-coloring of $G$. While there is a monochromatic triangle in $G$, it chooses one such triangle, and changes the color of a randomly chosen vertex of that triangle. Derive an upper bound on the expected number of such recoloring steps before the algorithm finds a 2-coloring with the desired property.

4. Let $G$ be a $d$-regular grapth. Show that the cover time of $G$ is $O(n^2 \log n)$. This shows that the cover times of graphs that are regular is smaller than cover times of graphs in which the vertex degrees exhibit a disparity (for example, the lollipop graph).

5. Let $\#F$ be the number of distinct satisfying truth assignments corresponding to a given DNF formula $F$. Consider the following alternate approach to devising an $(\varepsilon, \delta)$-FPRAS for estimating $\#F$. The $t^{th}$ trial of this algorithm consists of picking a random clause $C_t$, where the probability of choosing $C_j$ is proportional to the number of satisfying truth assignments for it. Next it selects a random satisfying truth assignment $a$ for $C_t$. Let $\text{cov}(a)$ be the set of

clauses that are satisfied by $a$. Define the random variable $X_t = 1/|\mathrm{cov}(a)|$. The estimator for $\#F$ is the random variable

$$Y = \eta \times \sum_{i=1}^{N} \frac{X_t}{N}$$

where $\eta = \sum_a \mathrm{cov}(a)$, where the sum is taken over all possible truth assignments $a$. Prove that $Y$ is a $(\varepsilon, \delta)$-approximation for $\#F$, when

$$N = \frac{cm}{\varepsilon} \ln \frac{1}{\delta}.$$

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