Homework 1
Due in class on September 20th

1. In class (and in Chapter 7 in Peleg) we saw how to distributively 3-color a rooted tree in $O(\log^* n)$ rounds. This problem asks you to describe a synchronous algorithm for 3-coloring a rootless (unrooted?), n-vertex tree in $O(\log n \cdot \log^* n)$ rounds.

   Your solution should consist of an algorithm (described in the style of Peleg), a proof of correctness showing that the algorithm produces a 3-coloring, and an analysis of the time complexity, showing that the algorithm runs in $O(\log n \cdot \log^* n)$ rounds. Use subroutines (as in Peleg) to improve the presentation of your algorithms and use lemmas to state and prove preliminary or intermediate results

   Hint: First prove that for some fixed fraction f, in any n-vertex tree, at least $f \cdot n$ of the nodes have degree at most 2.

2. A graph $G = (V, E)$ is *d-inductive* if every induced subgraph of G contains a vertex of degree at most d. It is well known that trees are 1-inductive and planar graphs are 5-inductive. It is also well known that a simple greedy algorithm suffices to $(d+1)$-color any d-inductive graph. Extend the algorithm from Problem 1 to d-inductive graphs. Specifically show that for any fixed d, there is a synchronous, distributed algorithm that colors any given d-inductive graph using $(d+1)$ colors, in $O(\log n \cdot \log^* n)$ rounds. This would, for example, imply that a planar graph can be 6-colored in $O(\log n \cdot \log^* n)$ rounds.

   For this problem, it will suffice just to mention how to modify the algorithm from Problem 1. Similarly, it will suffice just to indicate how the proof of correctness and the analysis from Problem 1 change.

3. Consider the following recurrence relation

   $$K_{i+1} = \Delta \cdot (\lceil \log K_i \rceil + 1),$$

   with $K_0 = \lceil \log_2 n \rceil$. Solve the recurrence to show that there is a $t = O(\log^* n)$ such that $K_t \leq 2\Delta$.

   You may recall that this recurrence appears in the analysis of the Cole-Vishkin algorithm for coloring bounded degree graphs. It represents the rate at which the number of bits used to represent the color of a node falls. Discussion of this appears on Page 84 in Peleg.

4. Provide a complete proof of Lemma 4.2 in Linial's paper. The proof of this lemma, as it appears in Linial's paper, simply asks that we use Example 3.2 from the Erdös-Frankl-Füredi paper. Show clearly how, using Example 3.2 with $d = 2$ yields Lemma 4.2.

5. Let $G = (V, E)$ be a UBG in a doubling metric space. For a vertex $v \in V$ suppose that for any two distinct neighors $x, y \in N(v)$, $d(x, y) \geq \Delta$. Derive an upper bound on the cardinality of $N(v)$ in terms of Δ and ρ (the doubling dimension of the metric space).