22C:153 Homework 3

Due: Tuesday, 5/13

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(1) Page 786 of your textbook contains an LP-formulation of the max-flow problem. One problem with this formulation is that it has $|V|^2$ variables and $\Theta(|V|^2)$ constraints, whereas it is possible to come up with an alternate LP-formulation with $|E|$ variables and $\Theta(|V| + |E|)$ constraints.

A simpler LP-formulation of the max-flow problem with fewer variables and fewer constraints, is given below. This formulation is obtained by adding a "dummy" edge (t, s) to the network with $c(t, s) = \infty$.

$$\max f(t, s)$$

subject to

$$\begin{aligned} \sum_{v:(u,v)\in E} f(u,v) - \sum_{v:(v,u)\in E} f(v,u) &= 0 \text{ for each } u \in V & (1) \\ f(u,v) &\leq c(u,v) \text{ for each } (u,v) \in E & (2) \\ f(u,v) &\geq 0 \text{ for each } (u,v) \in E & (3) \end{aligned}$$

(a) Consider the LP obtained by replacing constraint (1) above by the following:

$$\sum_{v:(u,v)\in E} f(u,v) - \sum_{v:(v,u)\in E} f(v,u) \leq 0 \text{ for each } u \in V$$

Show that the sets of feasible solutions to the two LPs are identical.

(b) Once we have made the modification in item (a), we have an LP in standard form and we can write down its dual program. Let $p(u)$, $u \in V$ be the dual variable associated with constraints of type (1) and $d(u, v)$, $(u, v) \in E$, be the dual variables associated with constraints of type (2). Write down the dual program for the above LP (after making the substitution described in item (a)).

(2) The dual program you have come up with for Problem 1 contains the non-negativity constraints:

$$p(u) \geq 0 \text{ for all } u \in V \text{ and } d(u,v) \geq 0 \text{ for all } (u,v) \in E.$$

Replace these constraints by the following 0-1 constraints:

$$p(u) \in \{0,1\} \text{ for all } u \in V \text{ and } d(u,v) \in \{0,1\} \text{ for all } (u,v) \in E.$$

Show that this 0-1 version of the dual program solves the *minimum s-t cut problem*.

The minimum s-t cut problem takes a directed edge-weighted graph $G = (V, E)$ and pair of distinguished vertices s and t and returns an s-t cut (a set of edges whose removal separates s and t) of minimum total weight.

(3) A square matrix B of integers is called *unimodular* if the determinant of B is in $\{-1, +1\}$. An integer matrix A is called *totally unimodular* if every square, nonsingular, submatrix of A is unimodular. Prove the following claim about totally unimodular matrices:

Claim: Let A be a matrix with elements in $\{-1, 0, +1\}$. A is totally unimodular if no more than two nonzero entries appear in any column, and if the rows of A can be partitioned into sets I_1 and I_2 such that:

(1) If a column has two entries of the same sign, their rows are in different sets;

(2) If a column has two entries of different signs, their rows are in the same set.

(4) It is well-known (and not too hard to show) that if A is an integer matrix that is totally unimodular and b is an integral vector, then any optimal solution of the following LPs is integral:

$$\text{minimize } c^T \cdot x \text{ subject to } A \cdot x \geq b,\ x \geq 0$$

$$\text{maximize } c^T \cdot x \text{ subject to } A \cdot x \leq b,\ x \geq 0$$

(a) Use this fact and the claim from Problem (3) to show that any optimal solution to the LP for max-flow (Problem 1(a)) is integral, provided all edge-capacities are integral.

(b) Similarly, show that any optimal solution to the dual LP (Problem 1(b)) is integral, provided all edge-capacities are integral.

(c) Use the above two results (items (b) and (c)) and the strong duality theorem for LPs to conclude the max-flow min-cut theorem for networks with integral edge-capacities.

(5) Suppose bus drivers are paid overtime for the time by which their routes in a day exceed t. Suppose there are n bus drivers, n morning routes with durations $x_1, x_2, \ldots, x_n$ and n afternoon routes with durations $y_1, y_2, \ldots, y_n$, and the objective is to assign one morning run and one afternoon run to each driver so as to minimize the total amount of overtime. Express this as a weighted matching problem, and prove that the best solution is to give the ith longest morning route and the ith shortest afternoon route to the same driver, for each i.

(6) Problem 34.5-1 (page 1017).
(**Hint:** There is a very easy reduction from an NP-complete problem that you are very familiar with.)

(7) Problem 34.5-2 (page 1017).

(8) Problem 34.5-5 (page 1017).

(9) Problem 34-3 (d), (e), and (f) (page 1019).

(10) Problem 34-4 (a) and (b) (page 1020).
