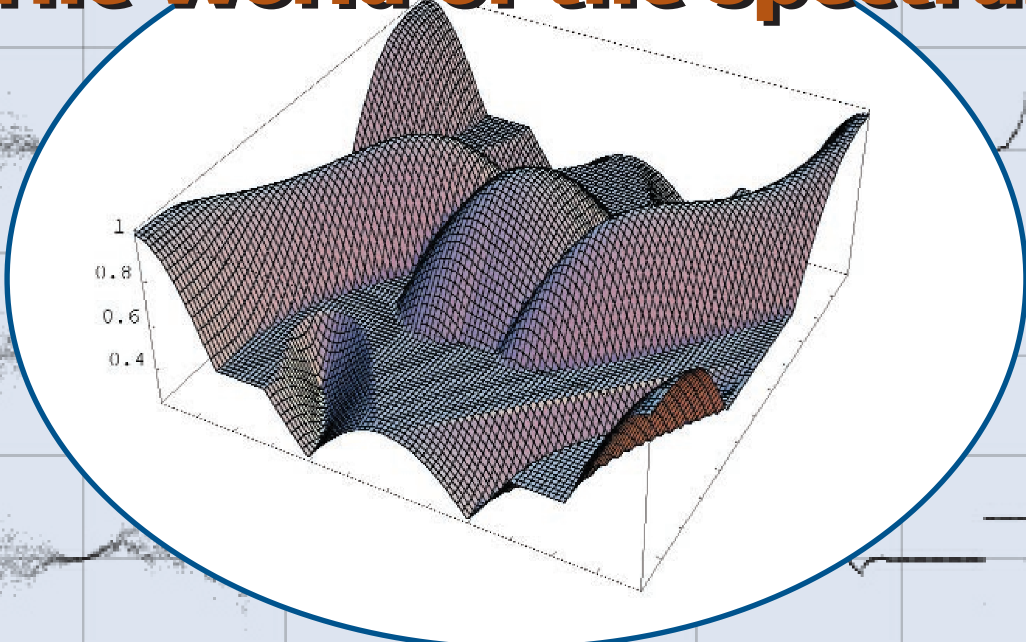
Applied and Numerical Harmonic Analysis

Wavelets Through A Looking Glass

The World of the Spectrum



O. Bratteli • P. Jorgensen

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Ola Bratteli
Palle Jorgensen

Wavelets Through a Looking Glass

The World of the Spectrum

with 147 illustrations by Brian Treadway

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Ola Bratteli
University of Oslo
Department of Mathematics
PB 1053 Blindern
Oslo N-0316
Norway
bratteli@math.uio.no
http://www.math.uio.no/~bratteli/

Palle Jorgensen
University of Iowa
Department of Mathematics
Iowa City, IA 52242-1419
U.S.A.
jorgen@math.uiowa.edu
http://www.math.uiowa.edu/~jorgen/

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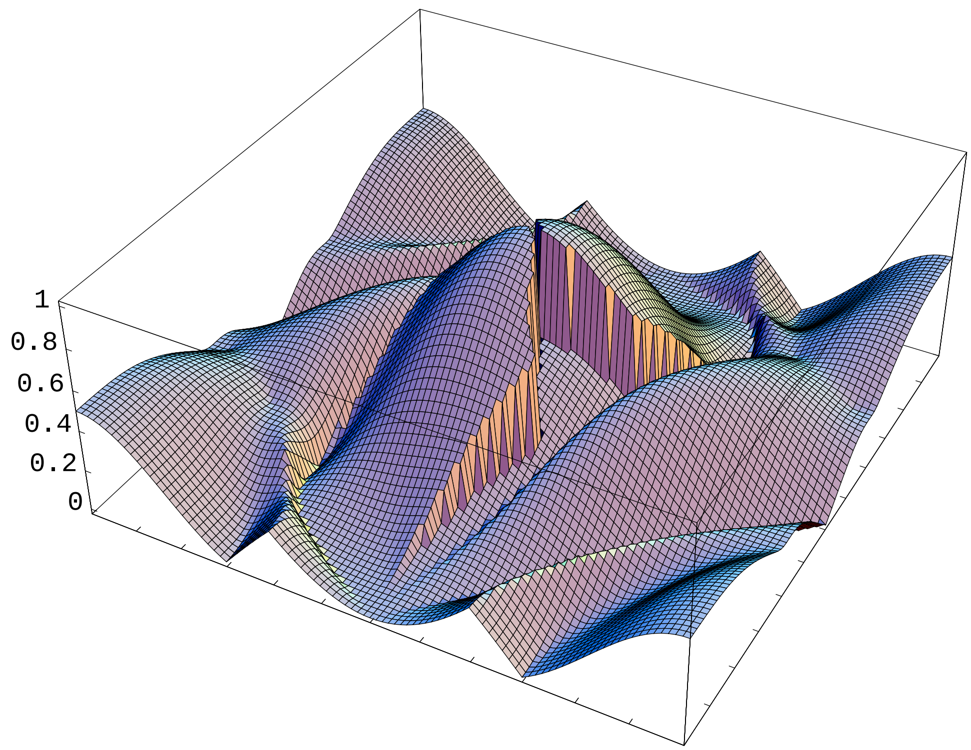
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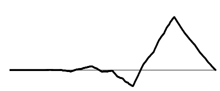
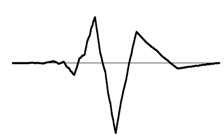
Preface

ADVANCES in communication, sensing, and computational power have led to an explosion of data. The size and varied formats for these datasets challenge existing techniques for transmission, storage, querying, display, and numerical manipulation. This leads to the paradoxical situation where experiments or numerical computations produce rich, detailed information, for which, at this point, no adequate analysis tools exist. —*Conference announcement, Joint IDR–IMA Workshop on Ideal Data Representation, Minneapolis, R. DeVore and A. Ron, organizers*





Wavelet theory stands on the interface between signal processing and harmonic analysis, the mathematical tools involved in digitizing continuous data with a view to storage, and the synthesis process, recreating, for example, a picture or time signal from stored data. The algorithms involved go under the name of filter banks, and their spectacular efficiency derives in part from the use of hidden self-similarity, relative to some scaling operation, in the data being analyzed. Observations or time signals are functions, and classes of functions make up linear spaces. Numerical correlations add structure to the spaces at hand, Hilbert spaces. There are operators in the spaces deriving from the discrete data and others from the spaces of continuous signals. The first type are good for computations, while the second reflect the real world. The operators between the two are the focus of the present monograph. Relations between operations in the discrete and continuous domains are studied as symbols. The mathematics involved in assigning operators to the symbolic relations is developed as a representation theory. The presentation is self-contained, and may serve as an introduction for readers who encounter these ideas for the first time and who would like to learn them from scratch.

A main point is the study of intertwining operators between, on the one side, the discrete world of high-pass/low-pass filters of signal processing, and on the other side, the continuous world of wavelets. There are significant issues in operator algebra and representation theory on both sides of the divide, and the intertwining operators shed light on central issues for wavelets in higher dimensions. Tools from diverse areas of analysis, as well as from dynamical systems and operator algebra, merge into the wavelet analysis. The diversity of techniques also adds to the charm of the subject, which continues to generate new mathematics.

The purpose of this book is twofold: first, to give a general presentation of some recent developments in wavelet theory, with an emphasis on techniques that are both fundamental and relatively timeless, and that have a geometric and spectral-theoretic flavor. It is our hope that it can be used equally well as a text for graduate students, as a reference book for specialists and researchers in neighboring fields, and in applications. Secondly, we are presenting some new results for the first time that have not previously appeared in papers, for example on the homotopy of multiresolutions, on approximation theory, and on the spectrum of associated transfer and subdivision operators. The backdrop to our book is Daubechies's classic [Dau92], but we also wish to stress the influence of a crucial paper of P. Auscher [Aus95] that solved two basic questions in wavelet theory, and that motivated the direction the subject has taken since then. The first question attacked in [Aus95] is about the limitations of the multiresolution method for wavelets: What are the wavelets that do not directly derive from a scaling function, or from some resolution subspace? The second question concerns localization in the dual variable, i.e., the frequency variable ω of the Fourier transform $\hat{\psi}(\omega)$ when ψ is given to be a wavelet function: Could $\hat{\psi}(\cdot)$ be supported in a half-line, $0 \leq \omega < \infty$? The answer turns out to be "no" unless $\hat{\psi}$ is a rather singular function. In summary, both questions are about frequency localization of wavelets, and our understanding of the tradeoffs between regularity and stability. Both questions are spectral-theoretic.

The interdependence among the chapters in this book can roughly be summarized as follows.

$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & 6 \\ 3 \arrow[r] & 4 \arrow[r] \arrow[ur] & 5 \end{tikzcd}$$

While this diagram does not exhaust all the interconnections among the chapters and the topics of the book, we hope it will assist instructors (and students) who might perhaps only need some, but not all, of the chapters in a semester-long course: An applied course that stresses wavelets and some of their applications could be based on the first row $1 \longrightarrow 2 \longrightarrow 6$, while the variants of the second row could be used alternatively in a course more focused toward operator theory. The tutorials at the start of each chapter supply further guides to interconnections of topics, and cross-fertilization among the various ideas and techniques that make up the book. Section 3.3 and the tutorial of Chapter 3 are somewhat independent of the other material in the lower row, and could beneficially be read in conjunction with Section 1.2 instead. In Section 2.5 we use some results from Sections 3.4, 4.3, and 4.4, but we do not otherwise need techniques from these sections here. Sections 1.3 and 1.4 are logically independent from the rest and may be omitted at the first reading. Likewise the results on homotopy of wavelets in Sections 2.1 and 2.4 may be omitted at the first reading except for the sequence (2.1.11)–(2.1.28) that is basic for Section 2.2. So in principle Chapters 3–5 (except Section 3.3) could be read as an introduction to transfer operators and their dual subdivision operators, independently of the wavelet applications of these results. Chapters 1–2 (with Section 3.3 added) could be read as an introduction to multiresolution wavelet theory.

Each chapter, and some sections within chapters, open with tutorials or primers of varying length. Written with minimal use of symbols and formulas, they serve both as summaries of some main ideas worked out in full detail inside the chapter (or section) in question, and also as guided tours through the background material, and especially as motivation. The tutorials are written in a style that is much more informal than that of the book proper, and this is intentional. They are in fact meant as friendly invitations to the topics to follow, with the emphasis on friendliness, even at the cost of occasional oversimplifications. The conclusions of tutorials and epigraphs are marked with "dingbats," typographic ornaments depicting scaling and wavelet functions. See Example 2.5.3 within for a discussion of how these depictions are generated. The ones used as dingbats are computed at the eighth cascade level. Brian Treadway is making available, on the authors' web sites (see note on next page), a sequence of such graphical depictions arranged to show how variations in the masking coefficients propagate and create a continuous moving picture in the variation of the scaling/wavelet functions, illustrating the algorithm in Chapter 1, (1.2.9)–(1.2.10), and a theorem in Chapter 2, Theorem 2.5.8.

Each chapter concludes with exercises. Most of them can readily be assigned as homework by an instructor teaching from the book, but a few exercises are more challenging, and they are marked with a star. Others will require the student to check journal articles, and do some research. They are marked with two stars. You will notice that Chapter 1 concludes with a relatively larger number of exercises than do the other chapters: Chapter 1 is where many basic concepts are first introduced. It is where the terminology for the rest of the book is discussed, and a number of the exercises are meant to help the student acquire a working familiarity with new terms and standard definitions. This is also why Chapter 1 concludes with a list of terminology. It turns out that some words are used differently by mathematicians, engineers, physicists, and computer scientists, and the list may perhaps serve as a dictionary. In fact, you might find it useful to consult this list right from the start, or the first time you come across a concept that you wish to have expounded.

You are invited to visit the World Wide Web pages of the authors* for updates and corrections to the book, for example concerning the open problem in Section 2.6.

A relatively moderate-sized book like this must of necessity omit many topics that are nonetheless both important and exciting. A list of exciting wavelet developments in the 1990s includes wavelet packets [CMW95, CoWi93], ridgelets and curvelets [CaDo00], the method of successive liftings for the discrete wavelet transform algorithm [DaSw98], [JelC01], applications to medicine and biology [AlUn96], and quantum computing wavelet programs [Kla99], [FiWi99], [Fre00]. While these are mentioned, or touched upon, inside the present book, they are treated only peripherally, as they branch off from the central theme of our book, and space is limited. Readers who may wish to look at the more advanced details of the exciting topics in the IMA workshop mentioned in the header of our Preface can consult the book [DLLP01]. It covers applications to three-dimensional computer graphics of the cascading refinement algorithm, and nonlinear wavelet approximations, among a list of current topics on the frontier of applied wavelet theory.†

The second named author (P.J.) thanks the Mathematics Institute of Oslo University for kind hospitality, and for support during a visit when the research was done. We also thank Brian Treadway for expert typesetting, graphics production, artistic and algorithmic creativity, corrections, *MATHEMATICA* work, helpful suggestions, and arbitration of disputes between the two authors. Among other things, Section 3.3 is almost entirely due to him. For mathematical help, we gratefully acknowledge kind suggestions from Erik Alfsen, Bill Arveson, Bachir Bekka (who provided the information in Section 1.8), David E. Evans, Peter Høyer, Gerald Kaiser, Andreas Klappenecker, David Kribs, David Larson, Roger D. Nussbaum, William L. Paschke, Jean Renault, Gilbert Strang, Brian Treadway, and Radka Turcajová. P.J. also acknowledges support from the Institute for Mathematics and its Applications in connection with the NSF-funded workshop in March 2001 mentioned above. P.J. had many discussions about the research of the present monograph with the other members of the workshop. Finally, we thank Gerald Kaiser for kindly letting us quote the paragraph which opens Section 1.1, from his book [Kai94]. In comparing our book with his, and others from that time, we note that the use of algorithms now has a more dominant role in wavelet analysis, a fact that is also reflected here. We are pleased to thank Ann Kostant for her encouragement from the outset, and her professional editorial advice and kind help during the final stages of the

* http://www.math.uio.no/~bratteli/wavebook/info.html and http://www.math.uiowa.edu/~jorgen/ .

† There are two lists of web addresses of researchers in wavelets and related fields that the reader may find useful: http://www.cs.wisc.edu/~amos/atpeople.html and http://www.cs.tamu.edu/faculty/klappi/people.html . A rich source of information on the latest developments in wavelet theory may be found in http://www.wavelet.org/wavelet/index.html .

preparation of this book. The editors John Benedetto and Akram Aldroubi of the ANHA series also have been a constant source of help and encouragement. P.J.'s view of the subject owes much to discussions he had with John and Akram at several conferences on wavelet analysis (see the text below), and the interaction that results from attending lectures from one or the other of us. We are grateful to reviewers for encouragement and detailed suggestions: they include A. Aldroubi, G. Strang, and J. Benedetto. We also benefited from a detailed list of helpful and constructive suggestions from an anonymous referee.

The material and the texture of this book grew out of recent developments in wavelet theory since the publication of [Dau92], and out of courses taught at the University of Iowa over six years. The latter include both advanced undergraduate courses and more specialized graduate courses. In addition, the authors have lectured at many universities in the U.S., Europe, and Asia on wavelets, for example, Georgia Institute of Technology (Atlanta), Louisiana State University, Wright State University (Ohio), Texas A & M University, University of Iowa, Vanderbilt University (Tennessee), University of Cincinnati (Ohio), University of Oslo (Norway), University of Aarhus (Denmark), University of Copenhagen (Denmark), University of Wales (U.K.), Imperial College (London), University of Rome I (Italy), Mathematisches Forschungsinstitut Oberwolfach (Germany), Universitatea Ovidius Constanta (Romania), Chinese University of Hong Kong, Hong Kong City University, Zhongshan University (Guangzhou, China), University of Shanghai (China), Australian National University, University of Newcastle (Australia), National University of Singapore, University of Toronto, and University of Waterloo (Canada). We are extremely grateful to the hosts at these universities, and to the students attending the lectures for the feedback they gave on some of the material going into the book. We wish to thank our hosts, Yang Wang, Chris Heil, and Jeff Geronimo (Georgia Tech.), Gestur Ólafsson (LSU), Steen Pedersen (Wright State U.), Dave Larson (Texas A & M), Akram Aldroubi, Doug Hardin, and Daoxing Xia (Vanderbilt U.), Klaus Thomsen (U. of Aarhus), David Evans (Cardiff University), Ka-Sing Lau (CUHK, Hong Kong), Judy Packer, Zuowei Shen, and S.L. Lee (Singapore), Derek Robinson (ANU, Australia), Ken Davidson (U. of Waterloo), and G.A. Elliott (U. of Toronto).

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Ola Bratteli and Palle E. T. Jorgensen

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