

Midterm II. (22C:19, Spring 2006)

Close books and notes, except one sheet of notes

Total points = 100

1. Matrices and Closures of Relations (20)

Given a $n \times n$ matrix M representing a relation $R \subseteq A \times A$, where $|A| = n$. Please describe how to check the following properties of R using M . You may describe your method verbally or use an algorithmic description. The *correctness* and *simplicity* of your methods decide the scores.

(a) (5) R is reflexive.

Answer: R is reflexive if for all $1 \leq i \leq n$, $M[i, i] = 1$.

(b) (5) R is symmetric.

Answer: R is symmetric if for all $1 \leq i < j \leq n$, $M[i, j] = 1$ whenever $M[j, i] = 1$.

(c) (5) R is antisymmetric.

Answer: R is antisymmetric if for all $1 \leq i < j \leq n$, $M[i, j] = 0$ whenever $M[j, i] = 1$.

(d) (5) R is transitive.

Answer: R is transitive if for all $1 \leq i, j, k \leq n$, $M[i, k] = 1$ whenever $M[i, j] = 1$ and $M[j, k] = 1$.

2. Functions (20)

Given two functions $f : \mathcal{R} \rightarrow \mathcal{R}$ and $g : \mathcal{R} \rightarrow \mathcal{R}$, where $f(x) = x^3 + x^2 - x - 1$ and $g(x) = 2^x$.

(a) (5) What is $f \circ g$?

Answer: $f \circ g(x) = f(g(x)) = (2^x)^3 + (2^x)^2 - (2^x) - 1$

(b) (5) What is $g \circ f$?

Answer: $g \circ f(x) = g(f(x)) = 2^{x^3+x^2-x-1}$.

(c) (5) Which of f and g is one-one? Justify your answer with evidence.

Answer: f is not one-one because $f(1) = f(-1)$. g is one-one because $2^x = 2^y$ implies $x = y$.

(d) (5) Which of f and g is onto? Justify your answer with evidence.

Answer: f is onto because for any $y \in \mathcal{R}$, there exists x such that $f(x) = y$. g is not onto because for $y < 0$, there is no x such that $g(x) = y$.

3. Cardinality of Sets (20)

Let $S = \{x \in \mathcal{R} \mid 0 < x < 2\}$. Show the sets S and \mathcal{R}^+ have the same cardinality.

Answer: Let us define $f : \text{cal}\mathcal{R}^+ \rightarrow S$, where $f(x) = 2/(1+x)$. Then f is one-one and onto. So the sets S and \mathcal{R}^+ have the same cardinality.

4. Binary Operations (20)

Please decide with evidence whether the groupoid $(\mathcal{R}, *)$, where $x * y = \lfloor x + y \rfloor$ for all $x, y \in \mathcal{R}$, is a semigroup.

Answer: $(\mathcal{R}, *)$ is not a semigroup because $(x * y) * z \neq x * (y * z)$ in general. For

example, when $x = y = 0.5$ and $z = 1$, then $(x * y) * z = \lfloor \lfloor x + y \rfloor + z \rfloor = 2$, but $x * (y * z) = \lfloor x + \lfloor y + z \rfloor \rfloor = 1$.

5. Congruences (20)

Please find all the solutions of the equation $13x \equiv 1 \pmod{100}$.

Answer: One solution of x in $\{0, \dots, 99\}$ is 77, which can be guessed by letting $x = 10a + b$, where a, b are digits. All the solutions of x can be expressed as $x = 77 + 100i$, where $i \in \mathcal{Z}$.