Rational Numbers & Periodic Decimal Expansions

A **rational number** is one that can be expressed as the ratio of two integers (i.e., whole numbers), for example, $\frac{1}{2}$ or $\frac{2}{3}$. A real number that is not a rational number is referred to as an **irrational number**.

Not all real numbers are rational — in fact, most are not. This is not immediately obvious since the rational numbers can easily serve many practical purposes. We formalize this claim as an assertion to be proven.

Assertion: The $\sqrt{2}$ is not a rational number.
Proof:
This is a proof by contradiction. That is, we assume that $\sqrt{2}$ *is* rational, and see that this inescapably leads us to a conclusion that is impossible, and that forces this assumption to be false. So we suppose that $\sqrt{2} = \frac{p}{q}$ for some integers p and q. Without loss of generality, we can also assume that p and q have no factors in common (since any common factors could be cancelled without changing the ratio). Now we do a little algebra, first squaring both sides of the equation to conclude that $2 = \frac{p^2}{q^2}$ or $p^2 = 2q^2$. But then since p^2 is even, it must be that p is also even (if p were odd, then p^2 is the product of two odds and must also be odd), say p = 2r. Now one more algebraic step, namely $p^2 = (2r)^2 = 4r^2 = 2q^2$, or $2r^2 = q^2$. Now we can again claim that since q^2 is even, it must be that q is even. But this means that p and q have the factor 2 in common contradicting our initial assumption. Hence no such integers p and q can exist to express $\sqrt{2}$ as a ratio, and so $\sqrt{2}$ is irrational.

Of course, this only establishes the existence of a single irrational number. But it is not difficult to repeat this argument to show *many* numbers are irrational. This can be done for other "roots", the mathematical constants π and e (base of natural logarithms), etc.

The main point in this note is to show there is a perfect correspondence between the rational numbers and the numbers with periodic or finite decimal expansions. That is, numbers such as $\frac{1}{3}$ have the unending, but repeating, decimal expansion .333 … . Often to provide a precise but succinct way to write such decimal expansions, the repeating part is written only once and marked with an overbar. For instance, $\frac{1}{3} = .\overline{3}$ and $\frac{3}{11} = .272727 \ldots = .\overline{27}$. Finite decimal expansions such as $\frac{1}{2} = .5$ could be regarded as repeating, where the repeating part is 0. Therefore they need no special attention.

Assertion: Each rational number has a periodic decimal expansion, and every number with a periodic decimal expansion is a rational number.
Proof:
This proof comes in two parts. First we see that each rational has a periodic decimal expansion. Then we show that every periodic decimal expansion can be expressed as the ratio of two integers.

<u>Part I</u>: each rational $\frac{p}{q}$ has either a finite or a periodic infinite decimal expansion.
This is evident if we visualize carrying out the familiar division algorithm. We illustrate this

on one specific example here, but the analysis is fully general. Consider carrying out the division $\frac{5}{12}$. This can be presented in the customary format as

```
        .4166 ...
     ____________
12 | 5.000
      48
      --
       20
       12
       --
        80
        72
        --
         80
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Notice that at this point the remainder of 8 has repeated. From this point on, this repetition will continue without end. Hence we have the repeating decimal expansion $.41\overline{6}$.

In general for a rational $\frac{p}{q}$, the remainder must be less than q. If a remainder of zero occurs, the process terminates with a finite expansion. Otherwise, after at most q steps a repetition of a non-zero remainder must occur. Once this happens we have an unending series of these repetitions giving an infinite repeating decimal expansion.

<u>Part II</u>: each number with a finite or periodic infinite decimal expansion is a rational number. For finite expansions, the result is immediate so we focus on infinite periodic expansions. To prove this case we need a couple of helping results. The first is about geometric sums.

Lemma 1: for any number $x \neq 1$ and any integer $n \geq 1$, $x+x^2+ \ldots +x^n = \frac{x-x^{n+1}}{1-x}$.
You are probably familiar with this result, and if not we will present its proof a little later in this course. The second result uses the first and concerns infinite geometric sums.
Lemma 2: for any number x whose absolute value is less than 1, $x+x^2+ \ldots +x^n + \ldots = \sum_{k=1}^{\infty} x^k = \frac{x}{1-x}$.
This result follows from Lemma 1 since the limit of x^k is 0 as k tends to infinity when $|x|<1$. This allows us to convert any periodic infinite decimal expansion into its rational number equivalent. For instance, $.\overline{14} = .141414 \ldots = 14/100 + 14/100^2 + 14/100^3 + \ldots =$ $14*(1/100 + 1/100^2 + 1/100^3 + \ldots) = 14*\left(\frac{1/100}{99/100}\right) = \frac{14}{99}$. If you're not convinced by this, try computing the decimal expansion of $\frac{14}{99}$.

We can conclude that *irrational* numbers have no natural finite representation — they have infinite non-repeating decimal expansions. The computing implications of this conclusion are that common calculations, even those starting with rational numbers, may easily lead to numeric results that have no representation that can be readily stored in a computer and are impossible to compute precisely!

Since many numbers (e.g., $\sqrt{2}$) cannot be computed exactly, we must often settle for approximations. For example, to compute the $\sqrt{a}$, we begin with an initial approximation

x_0, and then continue to obtain better approximations using the formula $x_{k+1} = \frac{1}{2} * (x_k + \frac{a}{x_k})$, for k = 0, 1, 2, …, until $|x_{k+1} - x_k| < \varepsilon$, where ε is the desired degree of accuracy. For instance, to obtain $\sqrt{2}$, take $x_0 = 1$, and then $x_1 = \frac{1}{2} * (1 + \frac{2}{1}) = 1.5$, $x_2 = \frac{1}{2} * (1.5 + \frac{2}{1.5}) = 1.41666\ldots$, $x_3 = \frac{1}{2} * (1.41667 + \frac{2}{1.41667}) = 1.414215\ldots$, etc. The development of such approximation techniques and the study of how rapidly they converge constitutes the field of numerical analysis. This area is concerned with the real (i.e., rational plus irrational) numbers, a set which is *not* discrete. Real numbers must be described by limiting techniques such as $e = \lim_{n \to \infty} (1 + \frac{1}{n})^n$ and $\pi = \lim_{n \to \infty} \sum_{k=0}^{n} \frac{8}{(4k+1)(4k+3)}$.