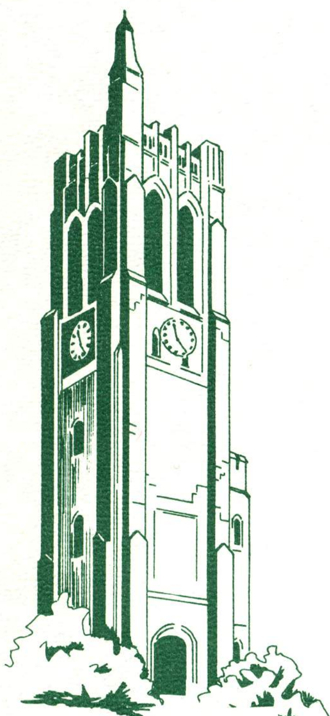
COMPUTER LABORATORY

Technical Bulletin

No. 16

STRUCTURE PRESERVING PROPERTIES OF CERTAIN CLASSES OF FUNCTIONS ON AUTOMATA

A. C. Fleck



MSU

Computer Laboratory
Michigan State University
East Lansing, Michigan 48823

INTRODUCTION

The effort to analyze computers has given rise to many ideas for a theoretical model of study. In terms of the amount of study given to such models, A. M. Turing's [1] has probably been the most productive. However, in the last few years the idea of a finite automaton has appeared more and more frequently in the literature. This is a model of a machine which accepts only tapes of finite length and which has only a finite number of (internal) states. The motivation of the study of finite automata is provided by the assumption that such a model is more realistic than, for instance, Turing's model.

In many of the studies of finite automata which have appeared, a method for transforming a given automaton into another automaton is introduced. We might cite as examples: reducing a machine to a machine with a minimal number of states [2], [3], raising a machine to a power [4], and forming the direct product of two automata [5].

This paper is devoted to a discussion of structure of automata. In particular, the problem of which structure properties are preserved under certain classes of transformations on automata is studied. It should be noted that, for the results of this paper, the set of states was not assumed to be finite. However, the results of this paper hold for finite automata and a few remarks are made concerning the application of the results to particular transformations.

The material following this introduction proceeds in four sections. In the first section the assumed definition of an automaton is dealt with. This definition is similar to that of Rabin and Scott [5], but is more general in several respects. The remainder of this section defines several structures on automata and explores their interrelations. It is also shown that a topology is naturally established on the set of states of each automaton.

The second section defines what is meant by a function on automata and, in particular, a continuous function. The structure preserving qualities of continuous functions are then investigated and are found to be extremely desirable.

The third section defines and studies a type of function called "operation preserving". It is shown that such functions have an even stronger structure preserving nature than continuous functions. It is also shown that a group, consisting of a particular set of functions, can be associated with each automaton.

The material of the last section is suggested by the result of the previous section which associates a group with each automaton. In particular, the results of this section lead to an answer for the question: when can the elements of the group associated with an automaton be expressed in terms of the next state function of the automaton?

SECTION 1

STRUCTURES AND A TOPOLOGY ON THE SET OF STATES

The definition of an automaton taken here parallels that of Rabin and Scott [5]. Occasionally a weighted, directed graph (state or transition diagram) will be used but only to specify an example. The explanation of this device is delayed until that time.

Definition 1.1 - An automaton, $A = (S, I, M, f)$, is a quadruple where S is a non-empty set (the set of states), I is a non-empty set (the set of inputs), M is a function (the next state function) taking $S \times I$ (Cartesian product) into S, f is a function (input composition) taking $I \times I$ into I such that (I, f) is a semi-group.

Definition 1.1 differs from the usual definition in several respects: first, the set of states is not assumed to be finite; second, the entire set of inputs is included directly in the quadruple; third, the input composition is arbitrary whereas it is usually assumed to be juxtaposition; lastly, an initial state and a set of final states are not specified since in the study of structure this is inessential information.

We now examine some structure properties of automata. Many of the structures defined below are discussed briefly in the literature but in most cases the properties have never been formally set down and their interrelations examined.

Definition 1.2 - A set of states, $T \subset S$, of an automaton $A = (S, I, M, f)$ is <u>open</u> if given any $s \in T$ and any $x \in I$, $M(s, x) \in T$.

Such a set is defined elsewhere in the literature as a stable set [6] or a submachine, but the term "open" is used here due to the topological nature and interpretation of the definitions and results to follow.

Theorem 1.1 - The union of arbitrarily many open sets of states of an automaton $A = (S, I, M, f)$ is an open set of states of A.

Proof: Set $U = \bigcup_K V_\alpha$ where $V_\alpha \subset S$ and V_α is open for all $\alpha \in K$ (index set). Then for $s \in U$, $s \in V_\alpha$ for some $\alpha \in K$. Then since V_α is open, $M(s,x) \in V_\alpha$ for all $x \in I$. Thus $M(s,x) \in U$ for all $x \in I$ and so U is open.

Theorem 1.2 - The intersection of arbitrarily many open sets of states of an automaton $A = (S, I, M, f)$ is an open set of states of A.

Proof: Set $U = \bigcap_K V_\alpha$ where $V_\alpha \subset S$ and V_α is open for all $\alpha \in K$ (index set). For any $s \in U$, $s \in V_\alpha$ for all $\alpha \in K$. Then since V_α is open for each $\alpha \in K$, $M(s,x) \in V_\alpha$ for all $x \in I$ and all $\alpha \in K$. Then $M(s,x) \in U$ for all $x \in I$ and U is open.

Theorem 1.3 - For any automaton $A = (S, I, M, f)$, the collection of open sets of states of A yields a topology on S, the set of states.

Proof: Obviously the null set, $\emptyset$, and the set S are open. This together with Theorems 1.1 and 1.2 establishes a topology [7].

Theorem 1.3 establishes all the structure results which hold for general topological spaces for automata in terms of the open sets of Definition 1.2. Thus we could, for example, follow the topological definition of limit state (point) and closed set and the usual well-known results would already be established.

Definition 1.3 - An automaton $A = (S, I, M, f)$ is <u>sequential</u> if $M(s, f(x, y)) = M(M(s,x), y)$ for all $s \in S$ and $x, y \in I$.

It should be noted that under the definition of an automaton by Rabin and Scott [5] (and similiar considerations due to Moore [3], Mealy [2], etc.) where the input composition is taken to be juxtaposition, the next state function is usually defined on a set of generators and then extended to the entire semi-group by means of the relation in Definition 1.3. Thus a "finite automaton" is usually considered to be sequential by definition. Definition 1.3 deserves one more comment. It will be seen that not only is sequentialness a natural concept, but for many of the results of this section and the next it is indeed necessary.

Definition 1.4 - An automaton $A = (S, I, M, f)$ is <u>strongly connected</u> if given any $s_1, s_2 \in S$, there exists an $x \in I$ such that $M(s_1, x) = s_2$.

The concept of strongly connectedness was first defined and investigated by Moore [3].

Theorem 1.4 - If an automaton $A = (S, I, M, f)$ is strongly connected, then there is no proper open subset of S.

Proof: Assume $U \subset S$ is a proper open subset. Then $S - U \neq \phi$. If $s_1 \in U$ and $s_2 \in S-U$, then $M(s_1, x) \in U$ for all $x \in I$ since U is open. But $s_2 \notin U$, hence $M(s_1, x) \neq s_2$ for all $x \in I$. Thus A is not strongly connected, a contradiction. Hence there is no proper open subset of S.

Lemma 1.1 - If an automaton $A = (S, I, M, f)$ is sequential, then for each $s \in S$, $T_s = \{s_1 | s_1 \in S, M(s, x) = s_1\}$ (i.e., the set of all s_1 such that $M(s, x) = s_1$ for some $x \in I$) is an open set.

Proof: Assume T_s is not open. Then there exists $s_1 \in T_s$ and $x \in I$ such that $M(s_1, x) = s_2 \notin T_s$. Now since $s_1 \in T_s$, $s_1 = M(s, y)$.

But then

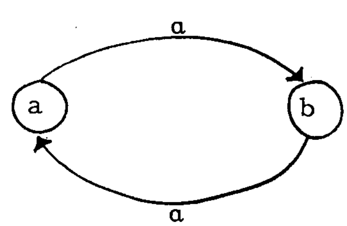
$M(s, f(y, x)) = M(M(s, y), x) = M(s_1, x) = s_2 \notin T_s$, a contradiction since $f(y, x) \in I$. Thus T_s is open.

Theorem 1.5 - If $A = (S, I, M, f)$ is a sequential automaton with no proper open subset of S, then A is strongly connected.

Proof: Assume A is not strongly connected. Then there exist $s_1, s_2 \in S$ such that $M(s_1, x) \neq s_2$ for all $x \in I$. Now by Lemma 1.1 $T_{s_1} = \{s \mid s \in S, M(s_1, x) = s\}$ is open. But $s_2 \notin T_{s_1}$. Hence T_{s_1} is a proper open subset (T_{s_1} is not empty since I is not empty), a contradiction. Thus A is strongly connected.

Two remarks are appropriate at this point: first, Theorems 1.4 and 1.5 can be stated in terms of a necessary and sufficient condition for strongly connectedness when the definition of an automaton assumes the property of sequentialness [6]. It will be necessary to use Theorem 1.4 frequently without the sequential property. Second, Theorem 1.5 is false if the sequential property is omitted as can be seen by the following example.

EXAMPLE I



$f(a, a) = a$

The device used to specify the automaton of Example I is called a state (or transition) diagram. Its meaning is:

the set of states, $S = \{a, b\}$, is the set of vertices of the graph;

the set of inputs, $I = \{\alpha\}$, is the set of weights of directed edges;

the next state function, $[M(a, \alpha) = b, \ M(b, \alpha) = a]$, is specified by the directed edges and their weights;

the input combination is specified in the margin (if not understood).

Notice that the automaton Example I is not strongly connected, but there is no proper open subset of states. This possible since the property of sequentialness is not present.

Definition 1.5 - An automaton $A = (S, I, M, f)$ is triangular if given any $s_1, s_2 \in S$, there exists $x, y \in I$ and $s \in S$ such that $M(s_1, x) = s = M(s_2, y)$.

Definition 1.6 - An automaton $A = (S, I, M, f)$ is not connected if there exist non-void, open sets $U, V \subset S$ such that $U \cup V = S$ and $U \cap V = \phi$; otherwise A is connected.

We mention that an automaton is connected if and only if its state diagram constitutes a connected graph.

Theorem 1.6 - If an automaton $A = (S, I, M, f)$ is triangular, then A is connected.

Proof: Assume A is not connected. Then there exist non-void, open $U, V \subset S$ such that $U \cup V = S$ and $U \cap V = \phi$. Now let $s_1 \in U$ and $s_2 \in V$. Then since A is triangular there exist $x, y \in I$ and $s \in S$ such that $M(s_1, x) = s = M(s_2, y)$. Now since $U \cap V = \phi$ and $U \cup V = S$ either $s \in U$ or $s \in V$. In the first case V is not open and in the second case U is not open, a contradiction. Thus A is connected.

We now introduced one more concept of structure and conclude the discussion of the interrelations arising.

Definition 1.7 - An automaton $A = (S, I, M, f)$ is <u>bi-connected</u> if whenever there exists an $x \in I$ such that $M(s_1, x) = s_2$, then there exists a $y \in I$ such that $M(s_2, y) = s_1$, where $s_1, s_2 \in S$.

This concept resembles closely that of strongly connectedness except that it is not assumed that a transition exists between every pair of states. However we have the following:

Theorem 1.7 - If $A = (S, I, M, f)$ is a sequential automaton, then a necessary and sufficnet condition that A be strongly connected is that A be connected and bi-connected.

Proof: (Sufficiency)

Suppose that A is not strongly connected. Then there exist $s_1, s_2 \in S$ such that $M(s_1, x) \neq s_2$ for all $x \in I$. Now by Lemma 1.1 $T_{s_1} = \{s | s \in S, M(s_1, x) = s\}$ is open.

But since A is bi-connected $S - T_{s_1}$ is also open. For suppose there exists $s \in (S - T_{s_1})$ such that $M(s, z) = t \in T_{s_1}$ for some $z \in I$. Then by bi-connectedness there exists $w \in I$ such that $M(t, w) = s$. But then T_{s_1} isnot open, a contradiction. Now T_{s_1} and $S - T_{s_1}$ are both open and T_{s_1} is not empty since I is not empty. Also $s_2 \in (S - T_{s_1})$ so $S - T_{s_1}$ is not empty. But we have $T_{s_1} \cup (S - T_{s_1}) = S$ and $T_{s_1} \cap (S - T_{s_1}) = \phi$. Thus A is not connected, a contradiction. Thus A is strongly connected.

The necessity of the condition is entirely obvious in the light of Theorem 1.4

Corollary 1.1 - If A = (S, I, M, f) is a sequential automaton, then a necessary and sufficient condition that A be strongly connected is that A be triangular and bi-connected.

Proof: Apply Theorems 1.6, 1.7 and the definitions.

Corollary 1.2 - If A = (S, I, M, f) is a sequential, bi-connected automaton, then the complement of every open set is open.

Proof: The proof of this fact is essentially the proof given in Theorem 1.7 to show T_{s_1} being open implies $S - T_{s_1}$ is open.

It is also interesting to note that sequentialness cannot be removed from the hypothesis of Theorem 1.7 for Example I provides a counter-example in this case.

SECTION 2

STRUCTURE PRESERVING PROPERTIES OF CONTINUOUS FUNCTIONS

In this section an investigation is begun of the relationship of the structure of a transformed automaton to the structure of the given automaton. In particular the concept of a continuous function of one automaton into another is defined and its structure preserving properties studied.

Definition 2.1 - For two automata, A = (S, I, M, f) and B = (T, J, N, g), by a function, h, of A into B, written $h:A \longrightarrow B$, is meant a function of S into T.

That is, a function on an automaton is merely a function on its set of states. For h, A and B as in Definition 2.1, the following usual notation will be used:

by the image, h(X), of a set $X \subset S$ under h is meant the set $h(X) = \{t | h(x) = t, x \in X\} \subset T$ and by inverse image $h^{-1}(Y)$, of a set $Y \subset T$ is meant the set $h^{-1}(Y) = \{s | s \in S, h(s) \in Y\} \subset S$.

Definition 2.2 - A function $h:A \longrightarrow B$, where A = (S, I, M, f) and B = (T, J, N, g), is <u>continuous</u> if for any open $Y \subset T$, $h^{-1}(Y) \subset S$ is open.

Or briefly; open sets come from open sets. The term continuous is chosen since Definition 2.2 is precisely the topological definition of a continuous function when A and B are topological spaces [7].

Definition 2.3 - A function $h:A \longrightarrow B$, where A = (S, I, M, f) and B = (T, J, N, g), is <u>open</u> if for any open $X \subset S$, $h(X) \subset T$ is open.

Theorem 2.1 - Let A = (S, I, M, f) be a strongly connected automaton and B = (T, J, N, g) be a sequential automaton. Then if $h:A \longrightarrow B$ is a continuous, onto function, B is strongly connected.

Proof: Assume B is not strongly connected. Then there exist $t_1, t_2 \in T$ such that $N(t_1, x) \neq t_2$ for all $x \in J$. Now by Lemma 1.1 $K_{t_1} = \{ t | t \in T, N(t_1, x) = t \}$ is an open set. Then since h is continuous $h^{-1}(K_{t_1}) \subset S$ is open. Then by Theorem 1.4 $h^{-1}(K_{t_1}) = S$, since A is strongly connected. Thus we have $h(S) = K_{t_1}$ and $t_2 \notin K_{t_1}$. But h was assumed to be onto, a contradiction. Hence B is strongly connected.

Corollary 2.1 - Let A = (S, I, M, f) be a sequential automaton and B = (T, J, N, g) be a strongly connected automaton. Then if $h:A \rightarrow B$ is an open, one-to-one, onto function, A is strongly connected.

Proof: Under the hypothesis $h^{-1}:B \rightarrow A$ is a continuous, onto function, so apply Theorem 2.1.

Before the next theorem is stated we must make reference to several well-known set equalities which hold for functions in general.

Let $f:S \rightarrow T$ be a function. Then the following statements hold:

(1) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$; $A, B \subset T$.

(2) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$; $A, B \subset T$.

(3) $f(A \cup B) = f(A) \cup f(B)$; $A, B \subset S$.

(4) $f(A \subset B) = f(A) \cap f(B)$; $A, B \subset S$.

Theorem 2.1 - Let A = (S, I, M, f) be a triangular automaton and B = (T, J, N, g) be a sequential automaton. Then if $h:A \rightarrow B$ is a continuous, onto function, B is triangular.

Proof: Assume B is not triangular. Then there exists $t_1, t_2 \in T$ such that $N(t_1, x) \neq N(t_2, y)$ for all $x, y \in J$. Certainly $t_1 \neq t_2$ or else

$N(t_1, x) = N(t_2, x)$ for all $x \in I$. Now by Lemma 1.1 the sets $K_{t_1} = \{t | N(t_1, x) = t\}$ and $K_{t_2} = \{t | N(t_2, x) = t\}$ are open and $K_{t_1} \cap K_{t_2} = \phi$. Now by Statement (2)

$$\phi = h^{-1}(\phi) = h^{-1}(K_{t_1} \cap K_{t_2})$$

$$= h^{-1}(K_{t_1}) \cap h^{-1}(K_{t_2}).$$

Let $K_1 = h^{-1}(K_{t_1})$ and $K_2 = h^{-1}(T_{t_2})$. Then we have $K_1 \cap K_2 = \phi$. Also since h is continuous K_1 and K_2 are open sets. Now K_{t_1} and K_{t_2} are not empty, since I is not empty and h is onto. So K_1 and K_2 are not empty. So let $s_1 \in K_1$ and $s_2 \in K_2$. Then since A is triangular, there exist $s \in S$ and $w, z \in I$ such that $M(s_1, w) = s = M(s_2, z)$. Now since $K_1 \cap K_2 = \phi$ either $s \in K_1$, $s \in K_2$ or $s \in (S - (K_1 \cup K_2))$. In the first case K_2 is not open, in the second K_1 is not open and in the last case neither K_1 nor K_2 is open, a contradiction in any circumstance. Thus B is triangular.

Corollary 2.2 - Let $A = (S, I, M, f)$ be a sequential automaton and $B = (T, J, N, g)$ be triangular. Then if $h: A \longrightarrow B$ is an open, one-to-one, onto function, A is triangular.

Proof: Under the hypothesis $h^{-1}: B \longrightarrow A$ is continuous and onto.

Theorem 2.3 - Let $A = (S, I, M, f)$ and $B = (T, J, N, g)$ be two automata and let A be connected. Then if $h: A \longrightarrow B$ is a continuous, onto function B is connected.

Proof: No proof of this theorem need be given here since all the concepts involved are topological in nature and the topological counterpart of this theorem is valid.

Corollary 2.3 - Let A = (S, I, M, f) and B = (T, J, N, g) be two automata and B be connected. Then if h:A⟶B is an open, one-to-one, onto function, A is connected.

Proof: As in Corollary 2.2.

Theorem 2.4 - If A = (S, I, M, f) is a strongly connected automaton and h is any function onto A, then h is continuous.

The proof of Theorem 2.4 is trivial in the light of Theorem 1.4 but we have the following interesting:

Corollary 2.4 - If h is any function of one automaton onto another which preserves strongly connectedness, then h is continuous.

Thus we have that the set of all functions on automata which preserve strongly connectedness is contained in the set of all onto, continuous functions. Moreover if we combine Corollary 2.4 with Theorem 2.1 we can make the following extremely important and desirable statement: For the set of all sequential automata, a necessary and sufficient condition that a function preserve strongly connectedness is that it be continuous.

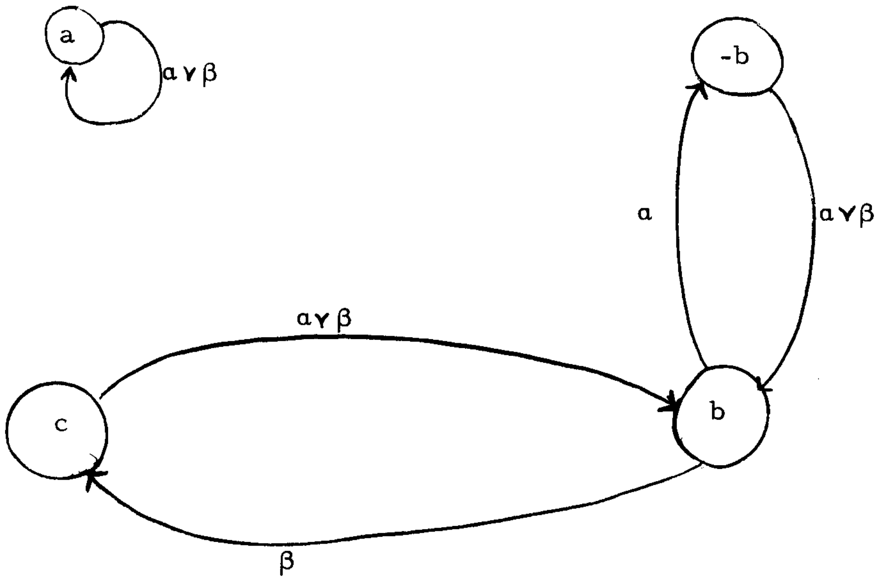
We have seen that continuous functions on automata have many desirable structure preserving properties. It is also interesting to notice that due to Theorem 1.3 we could have followed the topological definitions of a limit state (point) and a closed set of states and not only obtained the results pertaining to these concepts but also could have now given the usual necessary and sufficient condition(s) in terms of these concepts for a function to be continuous.

We also point out that for almost all the results where a continuous function preserves a structure it is necessary to the proof that the image automaton be sequential. It is easy to construct examples which show that with this restriction removed those theorems are in fact false. For instance, any function from a strongly connected (in fact any) automaton onto the automaton of Example I is necessarily continuous. However, recall that this automaton is not strongly connected. Also if our ideas were extended to models with outputs, both the reduction processes of Mealy [2] and Moore [3] would be continuous.

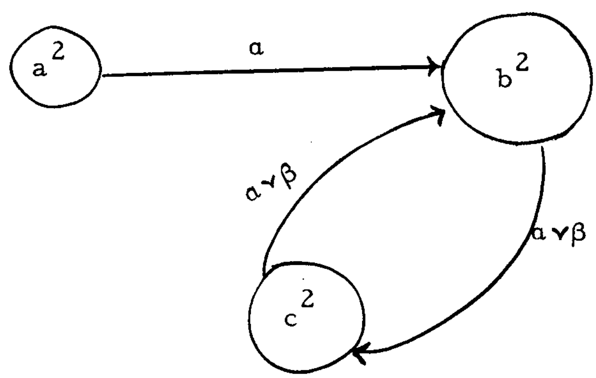
To conclude this section, we state an example which shows that the structure of bi-connectedness is not necessarily preserved by continuous functions.

EXAMPLE II

A



B



where the input composition is the usual juxtaposition (generators α, β) in both cases and thus A and B are sequential. Of course the transitions are not completely labeled and not all transitions are depicted. For instance in A, $M(-b, \beta\alpha) = -b$ and B, $M(a^2, \alpha\alpha) = c^2$.

Now we define $h: A \longrightarrow B$ by $h(x) = x^2$. Then it is easily checked that h is continuous and onto. However, A is bi-connected but B is not.

SECTION 3

STRUCTURE PRESERVING PROPERTIES OF OPERATION PRESERVING FUNCTIONS AND THE GROUP OF AN AUTOMATON

In this section we leave the general concept of continuous functions on automata and study a more specialized class of functions. We introduce the concept of operation preserving functions on automata and investigate their properties.

Definition 3.1 - If $h:A \rightarrow B$, where $A = (S, I, M, f)$ and $B = (T, I, N, g)$, satisfies $h[M(s,x)] = N(h(s),x)$ for all $s \in S$ and $x \in I$, then h is <u>operation preserving</u>. A concept similar to this, but for machines with outputs, has been briefly discussed by Ginsburg. [6]

We notice that Definition 3.1 applies only when A and B have semi-groups of inputs which are identified setwise. This restriction could be removed by establishing a correspondence between the input set of A and the input set of B (if they were different), but this complicates the discussion unnecessarily while yielding no significant refinement in the results.

Theorem 3.1 - If $h:A \longrightarrow B$, where $A = (S, I, M, f)$ and $B = (T, I, N, g)$ is operation preserving, then h is continuous.

Proof: Let $T_1 \subset T$ be open and let $s_1 \in h^{-1}(T_1) \subset S$. Then for each $x \in I$ consider $s_2 = M(s_1, x)$. $h(s_2) = h[M(s_1, x)] = N[h(s_1), x)] = t_1$.

Now since $h(s_1) \in T_1$ and T_1 is open $h(s_2) = t_1 \in T_1$. But then $s_2 \in h^{-1}(T_1)$ and thus $h^{-1}(T_1)$ is open and h is continuous.

Theorem 3.2 - If $h:A \longrightarrow B$, where $A = (S, I, M, f)$ and $B = (T, I, N, g)$, is operation preserving, then h is open.

Proof: Let $S_1 \subset S$ be open and $t_1 \in h(S_1)$ and $x \in I$ be arbitrary. Then $t_1 = h(s_1)$ for some $s_1 \in S_1$. Then since S_1 is open, $M(s_1, x) \in S_1$

for all $x \in I$ so we have $N(t_1, x) = N(h(s_1), x) = h[M(s_1, x)] \in h(S_1)$ for all $x \in I$. Thus $h(S_1)$ is open and so h is open.

Theorem 3.1 and Theorem 3.2 show that the class of all operation preserving function is a subclass of both the continuous and the open functions. Hence we may write, though space to do so will not be taken here, as corollarys to these theorems each of the results of Section 2 which deals with preservation of a structure by either a continuous or open function.

The following three theorems show that operation preserving functions have a much stronger structure preserving nature than continuous functions. In particular, Theorems 3.3 and 3.4 show that the restriction of sequentialness can be removed for operation preserving functions and Theorem 3.5 shows that bi-connectedness is preserved by operation preserving functions.

Theorem 3.3 - If $h: A \to B$, where $A = (S, I, M, f)$ and $B = (T, I, N, g)$, is an operation preserving, onto function and A is triangular, then B is triangular.

Proof: Let $t_1, t_2 \in T$. Then since h is onto there exists $s_1, s_2 \in S$ such that $h(s_1) = t_1$ and $h(s_2) = t_2$. Now since A is triangular there exists $x, y \in I$ and $s \in S$ such that $M(s_1, x) = s = M(s_2, y)$. But then $h[M(s_1, x)] = N[h(s_1), x] = N(t_1, x) = h(s) = h[M(s_2, y)] = N[h(s_2), y] = N(t_2, y)$. Hence B is triangular.

Theorem 3.4 - If $h: A \to B$, where $A = (S, I, M, f)$ and $B = (T, I, N, g)$, is an onto operation preserving function and A is strongly connected, then B is strongly connected.

Proof: Let $t_1, t_2 \in T$. Then since h is onto there exists $s_1, s_2 \in S$ such that $h(s_1) = t_1$ and $h(s_2) = t_2$. Then since A is strongly connected there exists $x \in I$ such that $M(s_1, x) = s_2$. But then $h[M(s_1, x)] = N(h(s_1, x) = N(t_1, x) = h(s_2) = t_2$. Thus B is strongly connected.

Theorem 3.5 - If $h: A \longrightarrow B$, where $A = (S, I, M, f)$ and $B = (T, I, N, g)$, is an onto, operation preserving function and A is bi-connected, then B is bi-connected.

Proof:

Let $t_1, t_2 \in T$ such that $N(t_1, x) = t_2$ for some $x \in I$. Then since h is onto there exists $s_1 \in S$ such that $h(s_1) = t_1$. But then for $s_2 = M(s_1, x)$, $h(s_2) = h[M(s_1, x)] = N(h(s_1), x) = N(t_1, x) = t_2$. Thus $h(s_2) = t_2$. Now A is bi-connected so there exists $y \in I$ such that $M(s_2, y) = s_1$ and then $t_1 = h(s_1) = h[M(s_2, y)] = N(h(s_2), y) = N(t_2, y)$. Thus B is bi-connected.

So we see operation preserving functions have a much stronger structure preserving nature than continuous functions. This idea is further emphasized by the following theorem which has no counterpart for continuous functions.

Theorem 3.6 - If $h: A \longrightarrow B$, where $A = (S, I, M, f)$ and $B = (T, I, N, g)$, is an onto, operation preserving function and A is sequential, then B is sequential.

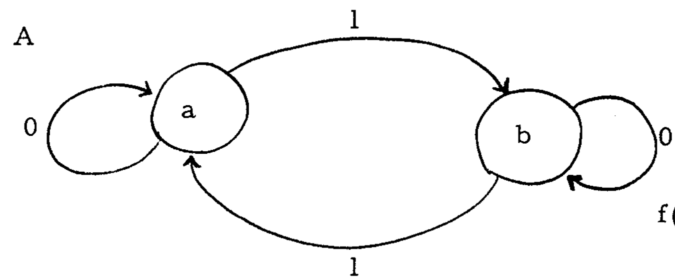
Proof: Since h is onto, for each $t \in T$ there exists $s \in S$ such that $h(s) = t$. Then

$$\begin{aligned} & N(t, f(x, y)) = N(h(s), f(x, y)) = h[M(s, f(x, y))] \\ = \quad & h[M(M(s, x), y)] = N(h[M(s, x)], y) \\ = \quad & N(N(h(s), x), y) = N(N(t, x), y) \end{aligned}$$

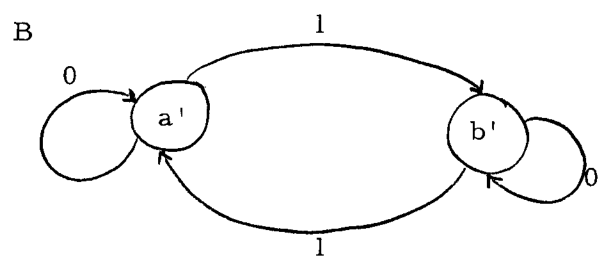
since h is operation preserving and A is sequential. Thus B is sequential.

It is interesting to notice that Theorem 3.6 is not true if the input compositions are distinct. The example below shows an operation preserving function carrying a sequential automaton onto a non-sequential automaton. Notice that the only thing that distinguishes the two automata is the input composition.

EXAMPLE III



$f(x, y) = x+y \pmod 2$



$g(x, y) = x \cdot y$

Define h(x) by $h(x) = x'$

Theorem 3.7 - The set of all functions $h:A \longrightarrow A$, where $A = (S, I, M, f)$, which are one-to-one, onto and operation preserving form a group.

Proof: The operation taken is the usual composition of functions. We have closure since if $h_1, h_2:A \longrightarrow A$ are 1-1, onto, operation preserving,

then h_1h_2 is 1-1, onto and $h_1h_2[M(s,x)] = h_1[M(h_2(s),x)] = M(h_1h_2(s),x)$ so h_1h_2 is operation preserving. Also it is well known that composition of functions is associative in general. Now the identity, $i(s) = s$, is certainly 1-1, onto and operation preserving, so it is in the set. It remains to show that inverses exist in the set. Suppose that $h:A \to A$ is 1-1, onto and operation preserving. Then certainly $h^{-1}:A \to A$ defined by $h^{-1}(x) = y$ if and only if $h(y) = x$ is 1-1 and onto and $hh^{-1}(s) = s = i(s)$. Now let $M(s,x) = s_1$ and $M(h^{-1}(s),x) = s_2$. Then
$h(s_2) = h[M(h^{-1}(s),x)] = M(hh^{-1}(s),x) = M(s,x) = s_1$. But since
$h(s_2) = s_1$, $h^{-1}(s_1) = s_2$. Thus $h^{-1}[M(s,x)] = h^{-1}(s_1) = s_2 = M(h^{-1}(s),x)$.
So h^{-1} is operation preserving and hence in the set and the proof is complete.

Theorem 3.7 is an extremely interesting result in that it associates with each automaton a group. In this connection we make

Definition 3.2 - For each automaton $A = (S, I, M, f)$ we denote by $G(A)$ the group associated with it by Theorem 3.7.

The general development to be followed now is suggested by the following interesting question: what relationships exist relating the structure of the automaton to the structure of the group associated with it?

After the results of Section four have established we will be able to make one statement in this regard, but otherwise such results are unknown to the author. However, the last results of this section relate to this question.

Theorem 3.8 - If $A = (S, I, M, f)$ is a strongly connected automaton, then $K[G(A)] \leq K[S]$ (where $K[X]$ denotes the cardinality of the set X).

Proof: Assume $K[G(A) > K[S]$ and let $s_1 \in S$ be any state. Consider the set $\{h(s_1)\}$ for all $h \in G(A)$. Since $K[G(A)] > K[S]$, there must exist distinct h_1, $h_2 \in G(A)$ such that $h_1(s_1) = h_2(s_1)$. Now let $s \in S$ be any state. Then since A is strongly connected there exists $x \in I$ such that $M(s_1, x) = s$. But then

$$h_1(s) = h_1[M(s_1, x)] = M(h_1(s_1), x) = M(h_2(s_1), x) = h_2[M(s_1, x)] = h_2(s).$$

Then since s is arbitrary $h_1 \equiv h_2$, a contradiction. Thus $K[G(A)] \leq K(S)$.

Corollary 3.1 - If $A = (S, I, M, f)$ is a strongly connected automaton and $h_1, h_2 : A \longrightarrow A$ are operation preserving and $h_1(s_0) = h_2(s_0)$ for some $s_0 \in S$, then $h_1 \equiv h_2$.

Proof: The proof of this somewhat more general statement is exactly the argument used in Theorem 3.8. Corollary 3.1 tells us that if $h \in G(A)$ and A is strongly connected, then h has no fixed points (unless h is the identity).

The statement of the next theorem is due to Weeg [8] and was first proven by him. A proof is given here for two reasons. First, Weeg's proof follows strictly from group theoretic arguments while the proof given here relies only on statements concerning the automaton. Second, the proof given here brings to light an interesting corollary not suggested by Weeg's proof.

Theorem 3.9 - If $A = (S, I, M, f)$ is a strongly connected automaton with n states, then the order of G(A) divides n.

Proof: First, by Theorem 3.7, G(A) is finite. Now suppose that $G(A) = \{h_1, h_2, \ldots, h_k\}$; $\{S = s_1 s_2, \ldots, s_n\}$ $(k \leq n)$ and consider the rectangular array of states

$$\begin{array}{cccc} h_1(s_1) & h_2(s_1) & \cdots & h_k(s_1) \\ h_1(s_2) & \cdots & & h_k(s_2) \\ \vdots & & & \vdots \\ h_1(s_n) & h_2(s_n) & \cdots & h_k(s_n) \end{array}$$

By Corollary 3.1 each row of this array consists of k distinct states. We show that each pair of columns either constitute disjoint sets or are identical (up to permutation).

Suppose $h_i(s_n) = h_j^{-h}(s_t)$. Then

$$h_i^{-1} h_i(s_u) = i\,(s_u) = s_u = h_i^{-1} h_j(s_t).$$ Let $h_i^{-1} h_j = h_m$. Then

$$i(s_u) = h_m(s_t)$$

$$h_1 i(s_n) = h_1(s_u) = h_1 h_m(s_t)$$

$$\vdots$$

$$h_k(s_u) = h_k h_m(s_t)$$

And of course $h_i h_m \neq h_j h_m$ for $i \neq j$ or else $h_i = h_j$. Hence any two columns are either completely disjoint or precisely the same set. Now of course all the states of A appear in the above array since some h_i is the identity. So if only the distinct columns are considered each state must appear precisely once. Thus if there are v distinct columns we have $k \cdot v = n$ or $k \mid n$.

We now write the essential statement made in Theorem 3.9 more concisely as Corollary 3.3. If A = (S, I, M, f) is a strongly connected automaton and

$$S = \{s_1, s_2, \ldots, s_n\} \text{ and } G(A) = \{h_1, h_2, \ldots, h_k\},$$

then for

$$S_i = \bigcup_{j=1}^{n} h_i(s_j), \quad i = 1, 2, \ldots, k.$$

We have

$$S_i \cap S_j = \phi \text{ or } S_i = S_j; \; i, j = 1, 2, \ldots k.$$

SECTION 4

REPRESENTATION OF THE GROUP ELEMENTS OF AN AUTOMATON BY ITS NEXT STATE FUNCTION

In this section we investigate further ideas suggested by the fact that a group is associated with each automaton. Now each element of the group of an automaton is a function from its set of states to its set of states. If we restrict the next state function to a single input symbol this is precisely the manner in which it maps. With this motivation in mind we now state the question which is answered in this section: when can the elements of the group of an automaton be expressed in terms of its next state function? To resolve this question we introduce, and give some investigation to, just one new concept.

Definition 4.1 - An automaton $A = (S, I, M, f)$ is abelian if for each $s \in S$ and $x, y \in I$, $M(s, f(x, y)) = M(s, f(y, x))$.

This structure seems somewhat artificial at first. However, it is an extremely powerful tool and does arise naturally in the light of the question to be answered in this section as is shown by

Theorem 4.1 - Let $A = (S, I, M, f)$ be a sequential automaton and $h \in G(A)$. Then if $h(s) = M(s, x_0)$ for some $x_0 \in I$, $M(s, f(x_0, y)) = M(s, f(y, x_0))$ for all $y \in I$ and $s \in S$.

Proof: Since h is operation preserving $h[M(s, y)] = M(h(s), y)$ for all $s \in S$, $y \in I$. But $h[M(s, y)] = M(M(s, y), x_0) = M(s, f(y, x_0))$ since A is sequential and $M(h(s), y) = M(M(s, x_0), y) = M(x, f(x_0, y))$ since A is sequential. Thus

$$M(s, f(y, x_0)) = M(s, f(x_0, y)).$$

In connection with Definition 4.1 we also notice that the automaton A is abelian if its input semi-group (I, f) is abelian. The only automaton exhibited thus far which is not abelian is automaton B of Example II.

Theorem 4.2 - If $h:A \to B$, where $A = (S, I, M, f)$ and $B = (T, I, N, f)$, is an onto operation preserving function and A is abelian, then B is abelian.

Proof: Let $t \in T$. Then since h is onto there exists $s \in S$ such that $h(s) = t$. Then $N(t, f(x, y)) = N(h(s), f(x, y)) = h[M(s, f(x, y))] = [M(s, f(y, x))]$ $= N(h(s), f(y, x)) = N(t, f(y, x))$ since A is abelian and h is operation preserving. Thus B is abelian.

We remark that it is easy to construct a counter example to Theorem 4.2 if the input compositions are allowed to be distinct. Also there is no statement for continuous functions corresponding to Theorem 4.2.

Theorem 4.3 - Let $A = (S, I, M, f)$ be an abelian, sequential automaton. Then for each $x_0 \in I$, h defined by $h(s) = M(s, x_0)$ is an operation preserving function of A into A.

Proof: Let $h(s) = M(s, x_0)$. Then $h[M(s, x)] = M(M(s, x), x_0)$

$= M(s, f(x, x_0))$ by sequentialness

$= M(s, f(x_0, x))$ by abelianness

$= M(M(s, x_0), x)$ by sequentialness

$= M(h(s), x)$.

Thus h is operation preserving.

The next theorem provides the answer for the question we posed at the beginning of this section. Namely, we give sufficient conditions for writing an element of the group of an automaton in terms of the next state function.

Theorem 4.4 - Let $A = (S, I, M, f)$ be an abelian, sequential, strongly connected automaton. Then for each $h \in G(A)$, $h(s) = M(s, x_0)$ for some $x_0 \in I$.

Proof: Let $h \in G(A)$ and $s_1 \in S$ and suppose $h(s_1) = s_2$. Then since A is strongly connected there exists $x_0 \in I$ such that $M(s_1, x_0) = s_2$. Let $h'(s) \equiv M(s, x_0)$. Then by Theorem 4.3 $h'(s)$ is an operation preserving function. But $h'(s_1) = h(s_1)$. Thus by Corollary 3.1 $h(s) \equiv h'(s) \equiv M(s, x_0)$.

Theorem 4.4 is an extremely powerful result. It presents us with the machinery to prove a result, as mentioned in section 3, which relates the structure of G(A) to the structure of A.

Theorem 4.5 - If $A = (S, I, M, f)$ is an abelian, sequential, strongly connected automaton, then G(A) is abelian.

Proof: Let $h_1, h_2 \in G(A)$. Then by Theorem 4.4, $h_1(s) \equiv M(s, x_1)$ and $h_2(s) \equiv M(s, x_2)$ for some $x_1, x_2 \in I$.

Now
$$\begin{aligned} h_1 h_2(s) &= M(M(s, x_2), x_1) \\ &= M(s, f(x_2, x_1)) \quad \text{by sequentialness} \\ &= M(s, f(x_1, x_2)) \quad \text{by abelianness} \\ &= M(M(s, x_1), x_2)) \quad \text{by sequentialness} \\ &= h_2 h_1(s). \end{aligned}$$

Thus $h_1 h_2 = h_2 h_1$ and G(A) is abelian.

An interesting (and open) question is the existance of other results of the form of Theorem 4.5. It would be particularly interesting if it could be shown that certain structure properties on G(A) force particular structure properties on A.

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