Propositional Logic

Definition: the (**well formed**) **formulas** (wffs) over a set of variables V consist of:
(i) **T**, **F**, and each X□V,
(ii) for each pair of wffs □ and □
- (□□),
- (□□□),
- (□ □),
- (□□ □),
- (□⇔□),

To avoid excessive parenthesizes in wffs we adopt the following precedence and take □ to be right-associative

highest precedence
□ □ □ , ⇔
lowest precedence

The definitions of the logical connectives are given by their "truth-tables"

P	Q	□P	P□Q	P Q	P□ Q	P⇔Q
T	**T**	**F**	**T**	**T**	**T**	**T**
T	**F**	**F**	**F**	**T**	**F**	**F**
F	**T**	**T**	**F**	**T**	**T**	**F**
F	**F**	**T**	**F**	**F**	**T**	**T**

Definition: an **assignment** to a set V of variables is a function □: V □ {**T**,**F**}. Each assignment is inductively extended to apply to wffs. For wffs □ and □
- □(□□) = □□(□),
- □(□□□) = □(□) □ □(□),
- □(□ □) = □(□) □(□),
- □(□□ □) = □(□) □ □(□),
- □(□⇔□) = □(□) ⇔ □(□), and
- □(**T**) = **T**, □(**F**) = **F**,

Definition: wffs □ and □ are **logically equivalent**, □ ≡ □, if □(□) = □(□) for each assignment □.

Definition: Let □ and □ be wffs. Then □ is **satisfiable** if there is an assignment □ so that □(□) = **T**; an unsatisfiable wff is also called a **contradiction**. If □(□) = **T** for every assignment □, then □ is a **tautology**.

Definition: a set of wffs $\mathcal{S}$ **logically implies** a wff □, $\mathcal{S}$ |= □, provided that for each assignment □ such that □(□) = **T** for each □□$\mathcal{S}$, □(□) = **T** (if $\mathcal{S}$ = ∅, write |= □ and □ is a tautology).

Definition: a **proof system** consists of the following constituents

- a subset of wffs called **axioms** — we expect there is a decision procedure to effectively determine whether or not a wff is an axiom.
- a finite collection $\{R_1, \dots, R_n\}$ of **inference rules**, where each rule R_i allows us to decide for wffs $\square_1, \dots, \square_{m_i}, \square$ whether or not $\square$ is a **direct consequence** of $\square_1, \dots, \square_{m_i}$, written $\dfrac{\square_1 \cdots \square_{m_i}}{\square}$.
- **proofs** which are sequences $\square_1, \dots, \square_k$ of wffs so that each i ($1 \le i \le k$), either $\square_i$ is an axiom or $\square_i$ is a direct consequence of some preceding wffs in the sequence; the last wff of a proof is called a **theorem**.

Two important rules of inference are:

- **modus ponens** — for any wffs $\square$ and $\square$, $\dfrac{\square \quad \square\square\ \square}{\square}$
- **resolution** — for any wffs $\square$, $\square$, and $\square$, $\dfrac{\square\ \square \quad \square\square\ \square}{\square\ \square}$.

Definition: in a proof system with axioms A, a wff $\square$ is a **consequence of a set of wffs** $\square$ if it is a theorem in the proof system with axioms A $\square$ $\square$. The elements of $\square$ are called **hypotheses** or **premises** and we write $\square \mid\!\!- \square$; if $\square = \varnothing$, write $\mid\!\!- \square$ (i.e., with no premises, the consequences are just the theorems).

Definition: a rule of inference $\dfrac{\square_1 \cdots \square_{m_i}}{\square}$ is **sound** provided that whenever each $\square_j$ ($1 \le j \le m_i$) is a tautology, $\square$ is also a tautology. A proof system is **sound** if each theorem is a tautology. A proof system is **complete** if each tautology is a theorem.